The Algebra of Physical Measurements

The Mathematical Algebra of René Descartes

Physics is deeply tied to mathematics but each has its own distinct algebra. Ever since Descartes introduced coördinate geometry, mathematics has been able to divorce its algebra from the need to represent physical weights and measures and deals solely in the domain of numbers. Modern mathematics, even with its explosion of abstractions, divorces itself from measured quantities. Descartes' form for a parabola $y = x^2$ deals solely in real numbers and requires a separate *coördinate basis* to map those numbers into the real world.

While it is possible to express physical laws and calculations within this mathematic framework, doing so throws away the constraints that guide physical intuition. Nevertheless we will still require some notation to bridge the gap between a variable representing a physical length and the number which can represent that length.

A Restricted Algebra of the Measurable

To this end let the variable ℓ be a distance without regards to the unit it is measured in; this names the distance but doesn't let it be used in any algebraic expression that a mathematician would understand. Let $1\,\mathrm{L}$ be a particular unit of length in its most abstract form. Now ℓ needs to be divided up into units of $1\,\mathrm{L}$ so the notation

$$\frac{\ell}{1\,\mathrm{L}}$$

is a very natural way to denote the variable ℓ as a number which can be manipulated algebraically. For simple linear motion, letting ℓ be a distance travelled, t an interval of time and K_v an arbitrary but fixed real number being a constant of proportionality relating the two

$$\frac{\ell}{1\,\mathrm{L}} = K_v \left(\frac{t}{1\,\mathrm{T}}\right) \qquad \Longleftrightarrow \qquad \frac{\ell/1\,\mathrm{L}}{t/1\,\mathrm{T}} = K_v \qquad \Longleftrightarrow \qquad \frac{\ell}{1\,\mathrm{L}} \propto \frac{t}{1\,\mathrm{T}} \qquad \Longleftrightarrow \qquad \ell \propto t$$

where the latter *varies-in-proportion-to* notation doesn't even make sense mathematically but in context is obvious. For simple accelerating motion starting from rest with a constant of proportionality K_a

$$\frac{\ell}{1\,\mathrm{L}} = K_a \left(\frac{t}{1\,\mathrm{T}}\right)^2 \qquad \Longleftrightarrow \qquad \frac{\ell/1\,\mathrm{L}}{(t/1\,\mathrm{T})^2} = K_a \qquad \Longleftrightarrow \qquad \frac{\ell}{1\,\mathrm{L}} \propto \left(\frac{t}{1\,\mathrm{T}}\right)^2 \qquad \Longleftrightarrow \qquad \ell \propto t^2$$

The final notation isn't mathematically sound but quite unambiguous.

An Example Double Proportion: Distance, Time and Speed

For simple linear motion, lets assume speed v is measurable on its own terms

$$\text{speed fixed } \frac{\ell}{1\,\mathrm{m}} \propto \frac{t}{1\,\mathrm{s}} \qquad \text{time fixed } \frac{\ell}{1\,\mathrm{m}} \propto \frac{v}{1\,\mathrm{mach}} \qquad \text{distance fixed } \frac{t}{1\,\mathrm{s}} \propto \left(\frac{v}{1\,\mathrm{mach}}\right)^{-1}$$

We can state that distance travelled is proportional to both time elapsed $\ell \propto t$ and the measured speed $\ell \propto v$ which we can write as $\ell \propto v \times t$ where the "$\times$" sign is part of the notation for a double proportion rather than as a multiplication of actual physical measurements. To explain what this actually means, if we hold speed constant and vary time elapsed then distance travelled varies in proportion; likewise, if we fix time elapsed and vary the speed then distance travelled varies in proportion. Imagining a rectangle with time elapsed along one side and speed along the other then distance travelled varying with the area of the rectangle satisfies our requirements; hence our adoption of the “times” sign. The units on the sides of the rectangle are irrelevant to the definition. When realized

$$\frac{\ell}{1\,\mathrm{mile}} \propto \left(\frac{v}{1\,\mathrm{mach}}\right)\left(\frac{t}{1\,\mathrm{hour}}\right) \qquad \text{or roughly at sea level} \qquad \frac{\ell}{1\,\mathrm{mile}} \approx 767\left(\frac{v}{1\,\mathrm{mach}}\right)\left(\frac{t}{1\,\mathrm{hour}}\right)$$

The approximate nature of the constant of proportionality is most unsatisfying. The speed of sound turns out to be a poor reference standard. We usually measure linear motion at a constant speed by seeing how far something travels in a unit of time or how long something takes to traverse a reference distance. In SI, the worldwide standard, that would be how far in metres (m) something travels in one second (1 s) for a measured velocity v in metres per second (${}^{\mathrm{m}}\!/_{\mathrm{s}}$). The measured pace p would be in units of seconds per metre (1 ${}^{\mathrm{s}}\!/_{\mathrm{m}}$). We can do this with great precision and express the relationship nicely in an equation form

$$\frac{\ell}{1\,\mathrm{m}} = \left(\frac{v}{1\,{}^{\mathrm{m}}\!/_{\mathrm{s}}}\right)\left(\frac{t}{1\,\mathrm{s}}\right) \iff \frac{{}^{\ell}\!/_{1\,\mathrm{m}}}{{}^{t}\!/_{1\,\mathrm{s}}} = \frac{v}{1\,{}^{\mathrm{m}}\!/_{\mathrm{s}}} \qquad \frac{t}{1\,\mathrm{s}} = \left(\frac{p}{1\,{}^{\mathrm{s}}\!/_{\mathrm{m}}}\right)\left(\frac{\ell}{1\,\mathrm{m}}\right) \iff \frac{{}^{t}\!/_{1\,\mathrm{s}}}{{}^{\ell}\!/_{1\,\mathrm{m}}} = \frac{p}{1\,{}^{\mathrm{s}}\!/_{\mathrm{m}}}$$

where

$$\left(\frac{v}{1\,{}^{\mathrm{m}}\!/_{\mathrm{s}}}\right)\left(\frac{p}{1\,{}^{\mathrm{s}}\!/_{\mathrm{m}}}\right) = 1$$

Despite having to deal with two different ways of measuring the same physical phenomenon, this is an extremely convenient way to express linear motion at a constant speed. It is conventional to express how fast something is moving by its *velocity*, the form of distance through a unit of time, and largely ignore measured *pace*. This is an unfortunate oversight as pace can often be more suited to a particular physics problem. This failing is not limited to equations as double proportions expresses with the *varies in proportion to* notation $\propto$ suffers the same shortcomings.

E.g. the Knot as a Derived Unit If the unit of length is a nautical mile and the unit of time is an hour then the unit of speed should be the nautical mile per hour. A nautical mile per hour is a *derived* unit in that it does not have a definition independent of other units. A nautical mile per hour is also called a *knot* based on the method of counting the knots on a rope attached to a *log* thrown overboard from a moving ship; the knots are counted until a sandglass empties and the log is retrieved. A sandglass always takes the same time to empty so there is no freedom in determining the distance between knots; the ratio of length between knots to a nautical mile shall be equal in proportion to the ratio of the time takes a sandglass to empty to an hour.

Eliminating Units from the Equations with Coherence

Much as we were able to unambiguously eliminate units from the proportionalities, in general, when the units are already matched

$$\frac{\ell}{1\,\bcancel{\mathrm{L}}} = \left(\frac{v}{1\,\frac{\bcancel{\mathrm{L}}}{\cancel{\mathrm{T}}}}\right)\left(\frac{t}{1\,\cancel{\mathrm{T}}}\right) \implies \ell = vt \qquad \frac{t}{1\,\bcancel{\mathrm{T}}} = \left(\frac{p}{1\,\frac{\bcancel{\mathrm{T}}}{\cancel{\mathrm{L}}}}\right)\left(\frac{\ell}{1\,\cancel{\mathrm{L}}}\right) \implies t = p\ell$$

The notation $\ell = vt$ and the notation $t = p\ell$ don’t make sense in terms of numbers but algebraically they seems fine. All the variables of kinematics can be seen as very straightforward products and ratios that mirror the definition of the units they are measured in; as such, the relations between them are most readily understood when presented in in equation form.

The mirroring of how mechanical variables are defined with the units they are measured in needs to be managed consistently and *coherently* to eliminate undesirable constants of proportionality appearing.

That is, there can be only a single defining path from the three sufficient base units to each derived unit. Consider a simple equation between power (P), torque (τ) and angular velocity (ω) in awkward units

$$\frac{P}{1\,\mathrm{hp}} = \frac{1}{5252}\left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{turn}}\!/_{\mathrm{min}}}\right)$$

The relationship is simple but the units are incoherent. Whereas in the coherent SI

$$\frac{P}{1\,\mathrm{W}} = \left(\frac{\tau}{1\,\mathrm{N{\cdot}m}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{rad}}\!/_{\mathrm{s}}}\right) \qquad \Longrightarrow \qquad P = \tau\omega$$

The unit 1 $^{\mathrm{rad}}\!/_{\mathrm{s}}$ doesn't seem to simplify out like the kinematic variables we saw before, but this is illusory; the moment arm which links angular torque to linear force also links the radius to circumferential motion — the radian is a dimensionless measure of angle which naturally arises and meshes beautifully with a coherent system of units. Without coherence we couldn't justify a simplified algebra; with coherence it is hard to imagine the alternative.

Unit Conversions We can recreate the power, torque and angular velocity relation with the awkward 5252 constant of proportionality by using the appropriate unit conversions from an underlying coherent system of units. Our coherent units will be based around minutes, feet and pounds-force; to be consistent with the moment definition of torque we must measure angles in radians (travel around the arc of a circle divided by the length of the moment arm)

$$\frac{P}{1\ {}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}} = \left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{rad}}\!/_{\mathrm{min}}}\right)$$

Conversions between commensurable units can be written with the fraction notation. The automotive definition of the horsepower is

$$\frac{1\,\mathrm{hp}}{1\ {}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}} = 33\,000 \qquad\qquad \frac{33\,000\ {}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}}{1\,\mathrm{hp}} = 1$$

The conversion from 1 turn to 1 rad is not "commensurable" by integer fractions but still understood

$$\frac{1\ {}^{\mathrm{turn}}\!/\cancel{{}_{\mathrm{min}}}}{1\ {}^{\mathrm{rad}}\!/\cancel{{}_{\mathrm{min}}}} = \left(\frac{1\,\mathrm{turn}}{1\,\mathrm{rad}}\right) = 2\pi \qquad\qquad \frac{2\pi\ {}^{\mathrm{rad}}\!/_{\mathrm{min}}}{1\ {}^{\mathrm{turn}}\!/_{\mathrm{min}}} = 1$$

From an equation $P = \tau\omega$ in coherent units of minutes, feet and pounds-force (and radians)

$$\begin{aligned}
\frac{P}{1\ {}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}} &= \left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{rad}}\!/_{\mathrm{min}}}\right) \\
\left(\frac{P}{1\ \cancel{{}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}}}\right)\left(\frac{33\,000\ \cancel{{}^{\mathrm{ft{\cdot}lbf}}\!/_{\mathrm{min}}}}{1\,\mathrm{hp}}\right) &= \left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ \cancel{{}^{\mathrm{rad}}\!/_{\mathrm{min}}}}\right)\left(\frac{2\pi\ \cancel{{}^{\mathrm{rad}}\!/_{\mathrm{min}}}}{1\ {}^{\mathrm{turn}}\!/_{\mathrm{min}}}\right) \\
\frac{P}{1\,\mathrm{hp}} &= \frac{2\pi}{33\,000}\left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{turn}}\!/_{\mathrm{min}}}\right) \\
\frac{P}{1\,\mathrm{hp}} &\approx \frac{1}{5252}\left(\frac{\tau}{1\,\mathrm{lbf{\cdot}ft}}\right)\left(\frac{\omega}{1\ {}^{\mathrm{turn}}\!/_{\mathrm{min}}}\right)
\end{aligned}$$

We multiplied the power term P by 1 then cancel out the old units to find power expressed in horsepower. Likewise for ω. This algebra is extremely convenient and just a little bit magical.

These automotive-friendly units are just weird. The FPS system of units, based around seconds, feet and pounds-force, are only a little more sensible with one horsepower equal to 550 footpounds per second. It is unfortunate that there are competing systems of coherent units which are clearly inferior to SI.

Long Variable Names

Having single letter variable names is the mathematical norm; however we might express linear motion as

$$\frac{\Delta\ell}{1\,\mathrm{L}} \propto \frac{\Delta t}{1\,\mathrm{T}}$$

where $\Delta\ell$ is an interval of length and Δt an interval of time. In standard mathematical usage, two-letter variables only occur in limited circumstance where the meaning of the initial letter is elided with certain *operators*: Δ for a difference between two endpoints and d for a differential (an infinitesimal difference at an instant) being the most prominent. This can help with readability, but only to a limited degree.

Outside of academia it is common to use long descriptive variable names. For this usage to be unambiguous it is necessary to distinguish between the next letter in a variable name and multiplication between juxtaposed variables. We use a centred dot to separate variable names wherever ambiguity might arise — which is anywhere that multi-character names might pop up. This is the same dot we have already seem when expressing numbers with in unit symbols (e.g. 1 lbf·ft or 1.3558 N·m) wherever the those symbols might run together ambiguously.

In physics it is far more common than in mathematics to use variable names based on a wider context. Mathematical variables pop into existence with little or no naming convention and their names lose significance once out of scope. The interpretation of a physical variable is tightly bound to its name. This is both a limitation and a potential source of ambiguity. Having a only fixed repertoire of one-character variable name can cause us to overload a name with superscripts, subscripts and other decorations to identify different variables with the same physical interpretation. And names are too scarce to be one-to-one with their interpretation; naming conflicts can occur when different domains overlap.

$$P = \tau\omega \qquad PV = nRT \qquad P = \textstyle\sum_i m_i v_i$$

Without context P could be power, pressure or momentum of a centre of mass; τ could be torque or proper time; V volume, voltage or potential energy; Having long variable names can help.

$$P = \tau \cdot \omega = \tau\omega \qquad \mathit{Pow} = \mathit{Tq} \cdot \mathit{Av} \qquad \mathit{Power} = \mathit{Torque} \cdot \mathit{AngularVelocity}$$

This is algebra and not a shorthand so would we need to name variables consistently while in scope; repeadtedly writing out whole word variable names seems counterproductive; variable names of a few characters seems better.

Monomials of a Real Multivariate Polynomial Domain for Representing Physical Measurements in a System of Coherent Units

Abstract algebra already provides the formalism for real numbers with coherent units, it just comes from an unexpected angle. Consider monomials of several variables; these are just special multivariate polynomials; those restricted to a single term. In abstract algebra a polynomial's interpretation as a real-valued function is ignored and they are treated as mathematical objects in their own right. Algebraically they are considered elements of a minimal extension of real numbers with one added free element for each free variable (i.e. each base unit corresponds to a free variable) and closed under addition, subtraction and multiplication but not division.

Monomials can be considered real numbers with base units raised to a non-negative power. Fractions of these lead to real numbers with units raised to any integer power — sufficient for what physicists need — extracting roots of measured values doesn't seem necessary. Having fractional exponents applied to the base units is unusual but does occur when deriving natural units. Simply replacing the base units with their square roots as the added free elements is sufficient for the particular algebraic formalism we require.

So we can multiply and divide numbers with units as we would expect, only adding or subtracting terms with matching units to stay in monomial form, and mathematics and physics stays in harmony, just not in the way we expected. The only limitation to this formalism is that it is wedded to a single system of coherent units where there is only one unit of a given dimensionality. In real-world use there are always conflicting units of the same dimensionality — conversion between them is made straightforward by the algebraic expression of how their base units interconvert. Extending the domain with several interconvertible units of like dimensionality should be possible and so recreating the algebra of numbers with units that we are familiar with — but this expanded formalism isn't mathematically interesting in any other context and doesn't appear in math textbooks.

Designing Coherent Systems of Units

SI: Coherent Second, Metre, Kilogram, Kelvin and Coulomb Units

Coherent systems of measurement are best characterized by their units for time, length, mass, temperature and charge. These five form the basis and all other units are derived from them. In SI these units are the second (s), metre (m), kilogram (kg), kelvin (K) and the coulomb (C) respectively.

time	length	speed	mass	impulse	force	work	action	temp.	charge	current
s	m	m·s^{-1}	kg	N·s kg·m·s^{-1}	N kg·m·s^{-2}	J kg·m^{2}·s^{-2}	J·s kg·m^{2}·s^{-1}	K	C	A C·s^{-1}

Note that SI is formally defined in terms of a slightly different set of base units together with another two fundamental units that we are ignoring, the candela (cd) and the mole (mol). Measures of angle and solid angle, as fractions of a circle or a sphere, are usually considered dimensionless; although, when they do appear in derived units they are usually expected to be treated as fundamental.

The base SI units are, from a modern perspective, quite arbitrary. The length of the day, the distance from the equator to the pole, the density of water, the freezing point of water, the boiling point of water at sea level — this isn't fundamental physics. The coulomb was defined via the ampere (A) in terms of the fundamental strength of the electromagnetic field but was done so in a most awkward fashion.

In modern SI the second, defined by the tick of an atomic clock, is so precise that the metre is now defined so that the speed of light is exactly $299\,792\,458$ m/s and the kilogramme is defined so that Planck's constant is exactly $6.626\,070\,15 \times 10^{-34}$ J·s. The kelvin is defined so that Boltzmann's Constant is exactly $1.380\,649 \times 10^{-23}$ J/K. The coulomb is now divorced from the other units and defined so that the elementary charge is exactly $1.602\,176\,634 \times 10^{-19}$ C. The measurements of these fundamental constants had become more precise than the old SI base units could represent demoting the metre, kilogramme and kelvin from their fundamental status. SI is now a well disguised system of natural units.

Natural Systems of Units

Rather than having all their base units be free to take on any value, in a system of natural units they are constrained by how certain fundamental constants of nature are presented. The system can choose from a gamut of constants: the universal constant of gravitation (G or its rationalized variant $\mathbb{G} \triangleq 4\pi\mathrm{G}$), the speed of light ($c$), Planck's constant ($h$ or *h-bar* $\hbar \triangleq h/2\pi$), Boltzmann's constant (k), the impedance of free space (Z_0), the charge on the positron (e) or the fine structure constant (α), the mass of the proton (m_p) or the electron (m_e); any subset of these are all possibilities. The derived system can be fully constrained, having no free base units at all, or partly constrained keeping one or more base units free and fundamental; keeping time as a fundamental unit will be explored here.

Symbols and Names for the Derived Units We will be Exploring

For our derived units we will set aside a few abbreviations and symbols which we can append together to make up a unit or system symbol. Unit symbols always include annotations for dimensionality; symbols for the entire system of units always omit it. Parts are always appended together in the order shown.

Common Scale	
charge	F
temperature	°
work	erg
force	dyn
mass	q
length	f
time	h

Uncommon Scale	
Planck scale	þ
1 second natural scale	ð
1 day natural scale	ðð

Radix†	
60	ſ
20	ϕ
12	ȸ
10	d
3	θ
2	b

†repeats

Dimensionality	
charge	q
temperature	a
work	e
force	f
mass	m
length	l
time	t

The scale of a system of units is either common or uncommon. Common scale units are given a name and a symbol to match and the system of units is identified solely by the radix part. Uncommon scale units are unnamed and only have the dimensionality suffix to distinguish between them.

Common Scale Unit Symbols† and Everyday Names

time	length	mass	force	work	temp.	charge
h⋯t	f⋯l	q⋯m	dyn⋯f	erg⋯e	°⋯a	F⋯q
heartbeat	neofoot	quatermass	neodyne	neoerg	neodegree	neofranklin

†wrapping a system abbreviation consisting of 1, 2 or 3 radix symbols

All Uncommon Scale Systems of Units

(þ)			
(ð)	(þſ)	(þd)	(þb)
(ðð)	(þß)	(þdd)	(þbb)

Common Scale Examples (d, dd, and ddd)

$$\begin{aligned} 1\,\text{Fdq} &= 1\,\text{Fddq} = 1\,\text{Fdddq} \\ &= 5.291\,925\,229\,\text{mC} \end{aligned}$$

$$\begin{aligned} 1\,\text{°da} &= 0.399\,67\,\text{K} \\ 1\,\text{qdm} &= 61.396\,\text{g} \\ 1\,\text{fdl} &= 0.572\,95\,\text{m} \\ 1\,\text{hdt} &= 1.911\,1\,\text{s} \end{aligned}$$

$$\begin{aligned} 1\,\text{°dda} &= 0.763\,823\,258\,\text{K} \\ 1\,\text{qddm} &= 117.336\,939\,\text{g} \\ 1\,\text{fddl} &= 0.299\,792\,458\,\text{m} \\ 1\,\text{hddt} &\triangleq 1\,\text{s} \end{aligned}$$

$$\begin{aligned} 1\,\text{°ddda} &= 0.884\,054\,697\,\text{K} \\ 1\,\text{qdddm} &= 135.806\,642\,\text{g} \\ 1\,\text{fdddl} &= 0.259\,020\,684\,\text{m} \\ 1\,\text{hdddt} &\triangleq \tfrac{24\times60\times60}{100\,000}\,\text{s} \end{aligned}$$

Uncommon Scale Examples (ð and ðð)

$$\begin{aligned} 1\,{}^{\text{ðe}}\!/\!_{\text{ða}} &\triangleq k \triangleq 1\,{}^{\text{ððe}}\!/\!_{\text{ðða}} \\ 1\,\text{ðe·ðt} &\triangleq \hbar \triangleq 1\,\text{ððe·ððt} \\ 1\,{}^{\text{ðl}}\!/\!_{\text{ðt}} &\triangleq c \triangleq 1\,{}^{\text{ððl}}\!/\!_{\text{ððt}} \end{aligned}$$

$$\begin{aligned} 1\,\text{ðl} &= 1\,\text{lightsecond} \\ 1\,\text{ððl} &= 1\,\text{lightday} \end{aligned}$$

$$\begin{aligned} 1\,\text{ða} &= 10^{-11}\,\text{°dda} \\ 1\,\text{ðm} &= 10^{-50}\,\text{qddm} \\ 1\,\text{ðl} &= 10^{9}\,\text{fddl} \\ 1\,\text{ðt} &\triangleq 1\,\text{s} = 1\,\text{hddt} \end{aligned}$$

$$\begin{aligned} 1\,\text{ðða} &= 10^{-16}\,\text{°ddda} \\ 1\,\text{ððm} &= 10^{-55}\,\text{qdddm} \\ 1\,\text{ððl} &= 10^{14}\,\text{fdddl} \\ 1\,\text{ððt} &\triangleq 1\,\text{day} = 10^{5}\,\text{hdddt} \end{aligned}$$

Planck Scale Examples (þ, þd, þſ and þb)

$$\begin{aligned} 1\,\text{þq} &= \sqrt{\hbar/Z_0} \\ &= 1\,\text{þſq} = 1\,\text{þdq} = 1\,\text{þbq} \\ &= 5.290\,817\,691 \times 10^{-19}\,\text{C} \end{aligned}$$

$$\begin{aligned} 1\,\text{þt} &= \sqrt{\hbar\mathbb{G}/c^5} \\ &= 1.911\,1 \times 10^{-43}\,\text{s} \\ &= 10^{-43}\,\text{hdt} \end{aligned}$$

$$\begin{aligned} 1\,\text{þda} &= 10^{43}\,\text{ða} \\ 1\,\text{þdm} &= 10^{43}\,\text{ðm} \\ 1\,\text{þdl} &= 10^{-43}\,\text{ðl} \\ 1\,\text{þdt} &\triangleq 10^{-43}\,\text{s} \\ &\equiv 10^{-43}\,\text{ðt} \equiv 10^{-43}\,\text{hddt} \end{aligned}$$

$$\begin{aligned} 1\,\text{þſt} &\triangleq 60^{-24}\,\text{s} \\ 1\,\text{þbt} &\triangleq 2^{-142}\,\text{s} \end{aligned}$$

Realizing Some Fanciful Mechanical Units

Deriving natural units for time, length and mass provides a basis sufficient for mechanics. In this respect, these three units seem more fundamental than those for temperature and charge.

Planck Units Denoting the units for time ($1\,\mathrm{T}$), length ($1\,\mathrm{L}$) and mass ($1\,\mathrm{M}$) we have the constraints $\mathrm{G} = 1\,\mathrm{M}^{-1}{\cdot}\mathrm{L}^{3}{\cdot}\mathrm{T}^{-2}$, $c = 1\,\mathrm{L}{\cdot}\mathrm{T}^{-1}$ and $h = 1\,\mathrm{M}{\cdot}\mathrm{L}^{2}{\cdot}\mathrm{T}^{-1}$ which wholly determine the $1\,\mathrm{T}$, $1\,\mathrm{L}$ and $1\,\mathrm{M}$

- called the *Planck time* $\approx 10^{-43}$ s, *Planck length* $\approx 10^{-34}$ m and *Planck mass* $\approx 10^{-8}$ kg respectively
- they are scaled to such extremes as to make them unsuitable as units for everyday use
- further requiring the constants to be identically one erases all dimensionality from the system
- G is only known with limited accuracy making the units useless for precision work

Regardless of how elegant they may be, the Planck units are a mere curiosity. Despite this, we will follow through as though it could lead to practical units.

A Rationalized Variant on the Planck Scale (þ) The rationalized constants $\mathbb{G}$ and $\hbar$ are more fashionable then their older cousins G and h; we would do well to constrain how the former are expressed rather than the latter.

Planck Units Scaled for Everyday Use For everyday use we would need to scale these three base units (denoted 1 þt, 1 þl and 1 þm for the rationalized variant) by large powers of ten to create a subsidiary basis ($1\,\mathrm{T}$, $1\,\mathrm{L}$ and $1\,\mathrm{M}$) upon which an plausible system could be built

$$\text{i.e. } \log_{10}\left(\frac{1\,\mathrm{T}}{1\,\text{þt}}\right),\ \log_{10}\left(\frac{1\,\mathrm{L}}{1\,\text{þl}}\right) \text{ and } \log_{10}\left(\frac{1\,\mathrm{M}}{1\,\text{þm}}\right) \text{ are integer exponents for scaling powers}$$

$$\text{with the effect that } \log_{10}\left(\frac{\mathbb{G}}{1\,\mathrm{M}^{-1}{\cdot}\mathrm{L}^{3}{\cdot}\mathrm{T}^{-2}}\right),\ \log_{10}\left(\frac{c}{1\,\mathrm{L}{\cdot}\mathrm{T}^{-1}}\right) \text{ and } \log_{10}\left(\frac{\hbar}{1\,\mathrm{M}{\cdot}\mathrm{L}^{2}{\cdot}\mathrm{T}^{-1}}\right) \text{ are integers}$$

These suffer from the exact same limits of precision as the underlying units. They are an impractical curiosity.

A Direct Derivation Natural-Enough for Everyday Use In deriving an everyday basis ($1\,\mathrm{T}$, $1\,\mathrm{L}$ and $1\,\mathrm{M}$) we can relax the constraints of the rationalized Planck units and only require $\mathbb{G}$, c and $\hbar$ be realized as powers of ten

$$\text{i.e. } \log_{10}\left(\frac{\mathbb{G}}{1\,\mathrm{M}^{-1}{\cdot}\mathrm{L}^{3}{\cdot}\mathrm{T}^{-2}}\right),\ \log_{10}\left(\frac{c}{1\,\mathrm{L}{\cdot}\mathrm{T}^{-1}}\right) \text{ and } \log_{10}\left(\frac{\hbar}{1\,\mathrm{M}{\cdot}\mathrm{L}^{2}{\cdot}\mathrm{T}^{-1}}\right) \text{ being integers}$$

These directly derived units ($1\,\mathrm{T}$, $1\,\mathrm{L}$ and $1\,\mathrm{M}$) are not necessarily scaled by powers of ten from the rationalized Planck units; they are instead *semicomensurable* with them

$$\text{i.e. } 2\log_{10}\left(\frac{1\,\mathrm{T}}{1\,\text{þt}}\right),\ 2\log_{10}\left(\frac{1\,\mathrm{L}}{1\,\text{þl}}\right) \text{ and } 2\log_{10}\left(\frac{1\,\mathrm{M}}{1\,\text{þm}}\right) \text{ are integers}$$

We have introduced enough wiggle room to devise conveniently sized units that are natural-enough for everyday use. But they are still totally impractical. But it does does open up room for a little speculation on the finer details of how practical units could be derived.

Everyday Systems of Natural Units: Denary, Binary, Sexagenary and Others

Restricting our scaling to powers of ten is needed because of our reliance on scientific notation and the underlying base ten numerals *(radix 10 — denary integers or decimal fractions).* But computers don't use scientific notation internally — floating-point numbers are the binary analogue to scientific notation and are based on binary numerals *(radix 2)* and powers of two. A conveniently sized system of fundamental units for computational purposes would be better served by scaling powers of two.

Restricting our scaling to powers of sixty just seems silly as such systems fell out of favour in the middle ages. All that is left of base sixty *(radix 60 — sexagenary integers or sexagesimal fractions)* in common usage is for minute and second fractions of the hour as a measure of time and of the degree as a measure of angle. But for everyday use, a system of units which embraces the hour, minute and second could be quite elegant.

Weird Ternary, Duodenary and Vicenary Units Base three *(radix 3 — ternary)* would seem a strange choice but balanced ternary may be the ideal representation of numbers for computational purposes, having excellent rounding behaviour and very simple arithmetic logic. Romans preferred base twelve fractions *(radix 12 — duodenary integers or duodecimal fractions)*, a bias that still shows up in traditional units. Base twenty *(radix 20 — vicenary integers or vigesimal fractions)*, on the other hand, is truly obsolete.

Overuse of, and Potential Ambiguity with, the Word "Base" A *base unit* is a part of the basis for a system of units. *Number base* is synonymous with the less common term *radix*. The *base of a logarithm*, the subscript to the log symbol, has no obvious synonym. In most discussions there would be a strong bias towards everyday decimal numbering; this exposition, however, breaks that particular rule.

Naming the Radix for Our Systems of Units

There are usually two synonymous names for the base of a number system coming from latin's distributive and ordinal numerals. *Denary* is named after the distributive numeral for ten and usually only applies to integral part of a number. *Decimal* is named after the ordinal numeral for ten and applies not only to the fractional part of a number but also the entire gamut of modern numbering (in particular to scientific notation).

radix	integral part	fractional part
b 2	binary	…
θ 3	ternary	…
d 10	denary	decimal
ȸ 12	duodenary	duodecimal
ϕ 20	vicenary	vigesimal
ſ 60	sexagenary	sexagesimal

Decimal is the more common name for base ten numerals; we will use the less common name to denote *denary units*; likewise for *duodenary units*, *vicenary units* and *sexagenary units. Binary units* and *ternary units* should be clear enough even without a well differentiated name.

Derived vs. Fundamental in a System of Units

We often call a unit *fundamental* if it part of the foundation for a whole system of coherent units, not that the measurement itself is inherently fundamental. In this sense the unit is free to be chosen arbitrarily and not bound within the constraints of the system itself. To disambiguate this usage of the word from measurements of the constants of nature, which are truly fundamental, we refer to the *base* units which comprise the *basis* for a system of units. In a formal sense, any unit subordinate in definition to the basis is often called *derived*, the value of such a unit bound to the values of the *base* units. This definition is too strict for us and we will use *fundamental* and *derived* in a more nuanced fashion. If we incorporate more physical laws into our choice of the base units themselves, we necessarily blur the line between derived and fundamental units. The speed of light (c) and Planck's constant (h or $\hbar$) are known to such high precision that, to ignore them, would be a disservice to any user of a system of units.

Derived vs. Fundamental: Not a Clean Division

In a natural system the difference between derived and fundamental units is a matter of degree; indeed, in a system of wholly derived units all measurements may be considered dimensionless. Using the standard abbreviations for dimensionality for the basis elements of time [T], length [L], mass [M], temperature [Θ] and charge [Q] we will name a system by appending all the base units together with more fundamental to the left in the manner TLMΘQ. Using Boltzmann's constant k and the impedance of free space Z_0 we can omit the fundamental units of temperature [Θ] and charge [Q] from the TLMΘQ basis to get an equivalent but formally incommensurable $\mathsf{TLM}(k \triangleq 1,\ Z_0 \triangleq 1)$ system (typically this system will be named simply TLM despite the potential ambiguity). Likewise, we can use the speed of light c and Planck's constant $\hbar$ to get $\mathsf{TM}(\mathrm{c} \triangleq 1)$ and $\mathsf{T}(\mathrm{c} \triangleq 1,\ \hbar \triangleq 1)$ systems.

basis	time	length	speed	mass	impulse	force	work	action	temp.	charge
TLMΘQ	T	L	LT^{-1}	M	MLT^{-1}	MLT^{-2}	$ML^{2}T^{-2}$	$ML^{2}T^{-1}$	Θ	Q
TLM									$ML^{2}T^{-2}$	$M^{1/2}LT^{-1/2}$
TM		T	1		M	MT^{-1}	M	MT	M	$M^{1/2}T^{1/2}$
T				T^{-1}	T^{-1}	T^{-2}	T^{-1}	1	T^{-1}	1

Within a term, units will be written in a reverse fashion with the more fundamental on the right and the more derived on the left. In the nineteenth century the fundamental order of base mechanical units was considered to be length, mass and time leading to the coherent metric systems named cgs (for centimetre-gramme-second) and MKS (for metre-kilogramme-second). Those names have stuck around although the order of units within terms almost always matches what is shown in the table above.

In the second-metre-kilogramme coherent system (SMK or, as traditionally named, MKS) the unit for impulse will be written as $1\,\mathrm{kg{\cdot}m{\cdot}s^{-1}}$ — in the second-centimetre-gramme coherent system (cgs) the unit would be $1\,\mathrm{g{\cdot}cm{\cdot}s^{-1}}$ — in a derivation of an intermediate natural system TLM where the units are named to match the standard abbreviations for dimensionality that would be $1\,\mathrm{M{\cdot}L{\cdot}T^{-1}}$ — and when we are simply trying to annotate dimensionality without regard to an actual unit we will use a sans-serif font in square brackets $[\mathsf{MLT^{-1}}]$.

Named units that are defined in terms of the base units are necessarily less fundamental than the base units themselves and you would expect to see them to the left of base units within a term; for impulse you might see the SI unit written as 1 N·s or a cgs unit written as 1 dyn·s. Note that there are no standard dimensionality abbreviations for units that would never be used as a base unit and you can't mix in a standard abbreviation that isn't a part of the basis for a unit you are annotating — so 1 N·s and 1 dyn·s still have dimensionality $[\mathsf{MLT^{-1}}]$. But in the obsolete variant of the FPS system, where the base units have dimensionality of time [T], length [L] and force [F], then 1 lbf·s of impulse would be annotated [FT]. No metric system has ever used force as a base unit for coherence

basis	time	length	speed	force	impulse	work	action	mass	density
TLF	T	L	LT^{-1}	F	FT	FL	FLT	$FL^{-1}T^{2}$	$FL^{-4}T^{2}$

There is little to recommend a TLF system. FPS in its modern (still obsolescent) incarnation, still wraps all its dynamic units around the pound-force but formally pretends it is derived from a base unit of mass ($1\,\mathrm{lbf} = 1\,\mathrm{slug{\cdot}ft{\cdot}s^{-2}}$) which it then replaces in common usage by the pound-mass ($1\,\mathrm{lbm} = \frac{1}{32.17405}\,\mathrm{slug}$).

Deriving Natural Units

For a derivation of $n+p$ base units from n constants of nature and p fundamental units, we can use a log transform to get a set of $n+p$ linear equations in $n+p$ variables. The solution to this is best described by an $n+p$ sized matrix inverse.

> For example, for a derivation of a natural TLMΘQ system we can constrain the results by expressing the form each chosen constant of nature can take; we name the row vector of exponents of the Q, Θ, M, L and T the dimensionality row vector. Stacking dimensionality row vectors one atop the other, one for each chose constants of nature, constructs a matrix. This is now a linear algebra problem.

Time Derived System of Units (given an arbitrary fundamental time T)

For the derivation of a system of natural units TLMΘQ from four constants of nature Z_0, k, $\hbar$ & c and one free fundamental unit of time T, the corresponding five-by-five matrix (of stacked dimensionality row vectors) is in upper-triangular form and, as such, has a very simple to compute matrix inverse also in upper-triangular form. Note that the linear equations can be considered to yield a general solution to the derivation with respect to the four constants of nature in terms of a single parameter T (italicized) which, incidentally, equals the arbitrary base unit of time 1 T (not italicized)

$$\begin{array}{rl} Z_0 &= 1\,\mathrm{Q^{-2}{\cdot}M{\cdot}L^{2}{\cdot}T^{-1}} \\ k &= 1\,\Theta^{-1}{\cdot}\mathrm{M{\cdot}L^{2}{\cdot}T^{-2}} \\ \hbar &= 1\,\mathrm{M{\cdot}L^{2}{\cdot}T^{-1}} \\ c &= 1\,\mathrm{L{\cdot}T^{-1}} \\ T &= 1\,\mathrm{T} \end{array} \Leftrightarrow \left.\begin{array}{rr} 1\,\mathrm{Q} = & \sqrt{\hbar/Z_0} \\ 1\,\Theta = & \frac{\hbar}{kT} \\ 1\,\mathrm{M} = & \frac{\hbar}{c^2T} \\ 1\,\mathrm{L} = & cT \\ 1\,\mathrm{T} = & T \end{array}\right\}\left\{\begin{bmatrix} -2 & 0 & 1 & 2 & -1 \\ & -1 & 1 & 2 & -2 \\ & & 1 & 2 & -1 \\ 0 & & & 1 & -1 \\ & & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ & & 1 & -2 & -1 \\ 0 & & & 1 & 1 \\ & & & & 1 \end{bmatrix}\right. \tag{1}$$

Note that the derived base unit of charge $1\,\mathrm{Q} = \sqrt{\hbar/Z_0}$ is independent of the parameter T. The base units for temperature and mass $1\,\Theta = \frac{\hbar}{kT}$ and $1\,\mathrm{M} = \frac{\hbar}{c^2T}$ vary in inverse proportion to T while the base units for distance $1\,\mathrm{L} = cT$ and time itself $1\,\mathrm{T} = T$ vary in direct proportion to the parameter T.

We can directly express any unit derived from the five base units in similar terms by multiplying the dimensionality row vector on the right by the five-by-five matrix inverse of (1) to yield exponents for Z_0, k, $\hbar$, c and T. For example, for some purely mechanical units where we can ignore the contributions of Z_0 and k,

$$\left.\begin{array}{rl} \text{speed unit} & 1\,\mathrm{L{\cdot}T^{-1}} \\ \hline \text{impulse unit} & 1\,\mathrm{M{\cdot}L{\cdot}T^{-1}} \\ \text{work unit} & 1\,\mathrm{M{\cdot}L^{2}{\cdot}T^{-2}} \\ \hline \text{unit for G} & 1\,\mathrm{M^{-1}{\cdot}L^{3}{\cdot}T^{-2}} \end{array}\right\} \left[\begin{array}{cc|ccc} & & 0 & 1 & -1 \\ \hline 0 & 0 & 1 & 1 & -1 \\ & & 1 & 2 & -2 \\ \hline & & -1 & 3 & -2 \end{array}\right] \left[\begin{array}{cc|ccc} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ \hline & & 1 & -2 & -1 \\ 0 & & & 1 & 1 \\ & & & & 1 \end{array}\right] = \left[\begin{array}{cc|ccc} & & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & -1 & -1 \\ & & 1 & 0 & -1 \\ \hline & & -1 & 5 & 2 \end{array}\right] \left\{\begin{array}{c} c \\ \hline \hbar c^{-1}T^{-1} \\ \hbar T^{-1} \\ \hline \hbar^{-1}c^{5}T^{2} \end{array}\right.$$

$$\text{i.e. for speed} \quad \left[\begin{array}{cc|ccc} \mathrm{Q} & \Theta & \mathrm{M} & \mathrm{L} & \mathrm{T} \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} Z_0 & k & \hbar & c & T \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \Rightarrow \quad 1\,\mathrm{L{\cdot}T^{-1}} = c$$

$$\text{for impulse} \quad \left[\begin{array}{cc|ccc} \mathrm{Q} & \Theta & \mathrm{M} & \mathrm{L} & \mathrm{T} \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} Z_0 & k & \hbar & c & T \\ 0 & 0 & 1 & -1 & -1 \end{array}\right] \quad \Rightarrow \quad 1\,\mathrm{M{\cdot}L{\cdot}T^{-1}} = \frac{\hbar}{cT}$$

$$\text{and for work} \quad \left[\begin{array}{cc|ccc} \mathrm{Q} & \Theta & \mathrm{M} & \mathrm{L} & \mathrm{T} \\ 0 & 0 & 1 & 2 & -2 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} Z_0 & k & \hbar & c & T \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \quad \Rightarrow \quad 1\,\mathrm{M{\cdot}L^{2}{\cdot}T^{-2}} = \frac{\hbar}{T}$$

We expressed the unit for speed only to have it be exactly c; similarly the unit of action will be constrained to be $\hbar$ by how the units were derived; the units with which k and Z_0 are expressed hold likewise. Being of known value they are necessarily independent of the parameter T.

Now consider the derived unit $1\,\mathrm{M^{-1}{\cdot}L^{3}{\cdot}T^{-2}}$ suitable for expressing the universal constant of gravitation

$$\left[\begin{array}{cc|ccc} \scriptstyle\mathrm{Q} & \scriptstyle\Theta & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{T} \\ 0 & 0 & \text{-}1 & 3 & \text{-}2 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} \scriptstyle Z_0 & \scriptstyle k & \scriptstyle\hbar & \scriptstyle c & \scriptstyle T \\ 0 & 0 & \text{-}1 & 5 & 2 \end{array}\right] \quad\Rightarrow\quad 1\,\mathrm{M}^{\text{-}1}{\cdot}\mathrm{L}^3{\cdot}\mathrm{T}^{\text{-}2} = \hbar^{\text{-}1}c^5T^2$$

The unit $1\,\mathrm{M}^{\text{-}1}{\cdot}\mathrm{L}^3{\cdot}\mathrm{T}^{\text{-}2}$ varies in proportion to the square of T and the universal constant of gravitation expressed in this system will, contravariant to the unit itself, vary inversely to the square of T. In particular, letting the rationalized universal constant of gravitation $\mathbb{G}$ be expressed in this system of units

$$\frac{\mathbb{G}}{1\,\mathrm{M}^{\text{-}1}{\cdot}\mathrm{L}^3{\cdot}\mathrm{T}^{\text{-}2}} = \frac{\mathbb{G}}{\hbar^{\text{-}1}c^5T^2} = \frac{\hbar\mathbb{G}}{c^5T^2} = \frac{\hbar\mathbb{G}/c^5}{T^2} = \left(\frac{\sqrt{\hbar\mathbb{G}/c^5}}{T}\right)^2 = \left(\frac{T}{\sqrt{\hbar\mathbb{G}/c^5}}\right)^{\text{-}2} \tag{2}$$

When $T = \sqrt{\hbar\mathbb{G}/c^5}$ then $\mathbb{G} = 1\,\mathrm{M}^{\text{-}1}\mathrm{L}^3\mathrm{T}^{\text{-}2}$ — this derivation of a unit of time $1\,\mathrm{T} = \sqrt{\hbar\mathbb{G}/c^5}$ is more properly considered as part of a wholly derived system of units with five constants of nature and no free parameter.

$T = \sqrt{\hbar\mathbb{G}/c^5}$ for Wholly Derived System of Units

For this derivation of a system of natural units TLMΘQ from five constants of nature (the previous four together with $\mathbb{G}$) and no other fundamental units (being wholly constrained), the corresponding five-by-five matrix differs from the upper-triangular matrix of the previous derivation in the bottom row only. It is no longer a simple upper-triangular matrix and is best considered a $1+1+3$ block upper-triangular matrix.

$$\left.\begin{array}{rl} Z_0 & = 1\,\mathrm{Q}^{\text{-}2}{\cdot}\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}1} \\ k & = 1\,\Theta^{\text{-}1}{\cdot}\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}2} \\ \hbar & = 1\,\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}1} \\ c & = 1\,\mathrm{L}{\cdot}\mathrm{T}^{\text{-}1} \\ \mathbb{G} & = 1\,\mathrm{M}^{\text{-}1}{\cdot}\mathrm{L}^3{\cdot}\mathrm{T}^{\text{-}2} \end{array} \Leftrightarrow \begin{array}{rl} 1\,\mathrm{Q} & = \sqrt{\hbar/Z_0} \\ 1\,\Theta & = \frac{1}{k}\sqrt{\hbar c^5/\mathbb{G}} \\ 1\,\mathrm{M} & = \sqrt{\hbar c/\mathbb{G}} \\ 1\,\mathrm{L} & = \sqrt{\hbar\mathbb{G}/c^3} \\ 1\,\mathrm{T} & = \sqrt{\hbar\mathbb{G}/c^5} \end{array}\right\}\left\{ \left[\begin{array}{cc|ccc} \text{-}2 & 0 & 1 & 2 & \text{-}1 \\ 0 & \text{-}1 & 1 & 2 & \text{-}2 \\ \hline & & 1 & 2 & \text{-}1 \\ & 0 & 0 & 1 & \text{-}1 \\ & & \text{-}1 & 3 & \text{-}2 \end{array}\right]^{\text{-}1} = \left[\begin{array}{cc|ccc} \text{-}1/2 & 0 & 1/2 & 0 & 0 \\ 0 & \text{-}1 & 1/2 & 5/2 & \text{-}1/2 \\ \hline & & 1/2 & 1/2 & \text{-}1/2 \\ & 0 & 1/2 & \text{-}3/2 & 1/2 \\ & & 1/2 & \text{-}5/2 & 1/2 \end{array}\right]\right. \tag{3}$$

This is identical to the previous system (1) with the pinned parameter $T = \sqrt{\hbar\mathbb{G}/c^5}$

We directly express any unit derived from the five base units by multiplying the dimensionality row vector on the right by the five-by-five matrix inverse of (3) to yield exponents for Z_0, k, $\hbar$, c and $\mathbb{G}$ respectively. For example, for some purely mechanical units

$$\left.\begin{array}{rl} \text{impulse unit} & 1\,\mathrm{M}{\cdot}\mathrm{L}{\cdot}\mathrm{T}^{\text{-}1} \\ \text{force unit} & 1\,\mathrm{M}{\cdot}\mathrm{L}{\cdot}\mathrm{T}^{\text{-}2} \\ \text{work unit} & 1\,\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}2} \\ \text{action unit} & 1\,\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}1} \end{array}\right\} \left[\begin{array}{c|ccc} & 1 & 1 & \text{-}1 \\ 0\,0 & 1 & 1 & \text{-}2 \\ & 1 & 2 & \text{-}2 \\ & 1 & 2 & \text{-}1 \end{array}\right] \left[\begin{array}{cc|ccc} \text{-}1/2 & 0 & 1/2 & 0 & 0 \\ 0 & \text{-}1 & 1/2 & 5/2 & \text{-}1/2 \\ \hline & & 1/2 & 1/2 & \text{-}1/2 \\ & 0 & 1/2 & \text{-}3/2 & 1/2 \\ & & 1/2 & \text{-}5/2 & 1/2 \end{array}\right] = \left[\begin{array}{c|ccc} & 1/2 & 3/2 & \text{-}1/2 \\ 0\,0 & 0 & 4 & \text{-}1 \\ & 1/2 & 5/2 & \text{-}1/2 \\ & 1 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \sqrt{\hbar c^3/\mathbb{G}} \\ c^4/\mathbb{G} \\ \sqrt{\hbar c^5/\mathbb{G}} \\ \hbar \end{array}\right.$$

$$\text{i.e. for impulse}\quad \left[\begin{array}{cc|ccc} \scriptstyle\mathrm{Q} & \scriptstyle\Theta & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{T} \\ 0 & 0 & 1 & 1 & \text{-}1 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} \scriptstyle Z_0 & \scriptstyle k & \scriptstyle\hbar & \scriptstyle c & \scriptstyle\mathbb{G} \\ 0 & 0 & 1/2 & 3/2 & \text{-}1/2 \end{array}\right] \quad\Rightarrow\quad 1\,\mathrm{M}{\cdot}\mathrm{L}{\cdot}\mathrm{T}^{\text{-}1} = \sqrt{\frac{\hbar c^3}{\mathbb{G}}}$$

$$\text{for force}\quad \left[\begin{array}{cc|ccc} \scriptstyle\mathrm{Q} & \scriptstyle\Theta & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{T} \\ 0 & 0 & 1 & 1 & \text{-}2 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} \scriptstyle Z_0 & \scriptstyle k & \scriptstyle\hbar & \scriptstyle c & \scriptstyle\mathbb{G} \\ 0 & 0 & 0 & 4 & \text{-}1 \end{array}\right] \quad\Rightarrow\quad 1\,\mathrm{M}{\cdot}\mathrm{L}{\cdot}\mathrm{T}^{\text{-}2} = \frac{c^4}{\mathbb{G}}$$

$$\text{and for work}\quad \left[\begin{array}{cc|ccc} \scriptstyle\mathrm{Q} & \scriptstyle\Theta & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{T} \\ 0 & 0 & 1 & 2 & \text{-}2 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} \scriptstyle Z_0 & \scriptstyle k & \scriptstyle\hbar & \scriptstyle c & \scriptstyle\mathbb{G} \\ 0 & 0 & 1/2 & 5/2 & \text{-}1/2 \end{array}\right] \quad\Rightarrow\quad 1\,\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}2} = \sqrt{\frac{\hbar c^5}{\mathbb{G}}}$$

But we already know the expression for action, not only because it is a defining constraint for this system of units, but because we knew from the time derived system which encapsulates this fully constrained system that action is defined independent of the T parameter and already fixed.

$$\text{i.e. for action}\quad \left[\begin{array}{cc|ccc} \scriptstyle\mathrm{Q} & \scriptstyle\Theta & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{T} \\ 0 & 0 & 1 & 2 & \text{-}1 \end{array}\right] \overset{(1)}{\mapsto} \left[\begin{array}{cc|ccc} \scriptstyle Z_0 & \scriptstyle k & \scriptstyle\hbar & \scriptstyle c & \scriptstyle\mathbb{G} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad\Rightarrow\quad 1\,\mathrm{M}{\cdot}\mathrm{L}^2{\cdot}\mathrm{T}^{\text{-}1} = \hbar$$

Relative Error in Conversions from SI (and with tunable constants)

SI	relative error	via (1)	relative error	via (3)	relative error
Z_0	1.50×10^{-10}	1 Q	7.50×10^{-11}	1 Q	7.50×10^{-11}
k	exactly	1 Θ	exactly*	1 Θ	1.10×10^{-5}
$\hbar$	exactly	1 M	exactly*	1 M	1.10×10^{-5}
c	exactly	1 L	exactly*	1 L	1.10×10^{-5}
$\mathbb{G}$	2.20×10^{-5}	1 T	exactly*	1 T	1.10×10^{-5}
		*whenever $\frac{T}{1\,\text{s}}$ is exact			

$$
\begin{aligned}
k &= 13.806\,49 \times 10^{-24}\,{}^{\text{J}}\!/\!_{\text{K}} \\
h &= 662.607\,015 \times 10^{-36}\,\text{J}\cdot\text{s} \\
c &= 299\;792\,458\,{}^{\text{m}}\!/\!_{\text{s}} \\
Z_0 &= 376.730\,313\,412\,{}^{\text{J}\cdot\text{s}}\!/\!_{\text{C}^2} \\
&\ \pm 0.000\,000\,059\,{}^{\text{J}\cdot\text{s}}\!/\!_{\text{C}^2}\ (\Omega) \\
G &= 66.743\,0 \times 10^{-12}\,{}^{\text{N}\cdot\text{m}^2}\!/\!_{\text{kg}^2} \\
&\ \pm 0.001\,5 \times 10^{-12}\,{}^{\text{N}\cdot\text{m}^2}\!/\!_{\text{kg}^2}
\end{aligned}
$$

A Conversion from SI to a natural basis using the (1) transformation relies on the four constants Z_0, k, $\hbar$ and c and the on arbitrary T which will usually be defined such that $\frac{T}{1\,\text{s}}$ is exact. Because of how SI has been defined since 2019 all the derived base units with the exception of the unit of charge will be exactly convertible to SI. This holds even if we treat the Z_0, k, $\hbar$ and c as tunable parameters to the (1) transformation where we can divide them by selected powers of a number base in order to size the resultant derived basis units. We'll do much the same for the T parameter later on when we want the option of our derived units to reference a standard day of 85400 s while still being sized to being on the order of a *heartbeat*. This sort of tuning, being an exact division, has no effect on error terms.

A Conversion from SI to a natural basis using the (3) transformation relies on the five constants Z_0, k, $\hbar$, c and $\mathbb{G}$ where only the middle three have an exact definitions. The (3) transformation can be considered can be considered a special case of the (1) transformation where the T parameter is set to $\sqrt{\hbar\mathbb{G}/c^5}$. Because the derived unit of charge is independent of T its conversion from SI retains it's small relative error; but the limited precision of $\mathbb{G}$ ensures that conversions for 1 T as well as all the remaining base units also lacks precision; and tuning for size can make no difference.

Practically-Sized Units from Improbably-Sized Planck Scale Units

Improbably-Sized Planck Scale Units (þ, þf, þd, þb, þß, þdd and þbb)

With no radix abbreviation (þ) the derived Planck scale units will be wholly determined by the (3) transformation or, equivalently, by the (1) transformation via a T parameter that is exactly the rationalized Planck time

$$1\,\text{þt} = \sqrt{\hbar\mathbb{G}/c^5} \approx 1.911 \times 10^{-43}\,\text{s}$$

With one (þf, þd and þb) or two (þß, þdd and þbb) radix abbreviations the Planck scale units will be derived from the (1) transformation via a parameter T which will only approximate 1 þt

$\overset{(1)}{\mapsto}$ parameter T	sexagenary	denary	binary
$\frac{1\,\text{s}}{1\,\text{þt}} \approx \frac{1\,\text{s}}{T} =$	$\frac{1\,\text{s}}{1\,\text{þft}}$ a power of sixty (þf)	$\frac{1\,\text{s}}{1\,\text{þdt}}$ a power of ten (þd)	$\frac{1\,\text{s}}{1\,\text{þbt}}$ a power of two (þb)
$\frac{1\,\text{day}}{1\,\text{þt}} \approx \frac{1\,\text{day}}{T} =$	$\frac{1\,\text{day}}{1\,\text{þßt}}$ a power of sixty (þß)	$\frac{1\,\text{day}}{1\,\text{þddt}}$ a power of ten (þdd)	$\frac{1\,\text{day}}{1\,\text{þbbt}}$ a power of two (þbb)

The units þft, þdt, þbt, þßt, þddt and þbbt that approximate 1 þt will have an exact definition in seconds while the original 1 þt can only be known with limited precision (cf. $\sqrt{\mathbb{G}}$). For a high-precision example

$$
\begin{array}{llclcl}
3.767\,303\,134 \times 10^{2}\,{}^{\text{J}\cdot\text{s}}\!/\!_{\text{C}^2} & = Z_0 & = 1\,{}^{\text{þdde}\cdot\text{þddt}}\!/\!_{\text{þddq}^2} & & 1\,\text{þddq} & = 5.290\,817\,692 \times 10^{-19}\,\text{C} \\
1.381\,49 \times 10^{-23}\,{}^{\text{J}}\!/\!_{\text{K}} & = k & = 1\,{}^{\text{þdde}}\!/\!_{\text{þdda}} & & 1\,\text{þdda} & = 8.840\,546\,970 \times 10^{31}\,\text{K} \\
1.054\,571\,818 \times 10^{-34}\,\text{J}\cdot\text{s} & = \hbar & = 1\,\text{þdde}\cdot\text{þddt} & \overset{(1)}{\Rightarrow} & 1\,\text{þddm} & = 1.358\,066\,426 \times 10^{-9}\,\text{kg} \\
2.997\,924\,58 \times 10^{8}\,{}^{\text{m}}\!/\!_{\text{s}} & = c & = 1\,{}^{\text{þddl}}\!/\!_{\text{þddt}} & & 1\,\text{þddl} & = 2.590\,206\,837 \times 10^{-35}\,\text{m} \\
0.864\,\text{s} & = T & = 1\,\text{þddt} & & 1\,\text{þddt} & = 8.640\,000\,000 \times 10^{-44}\,\text{s}
\end{array}
$$

★ $T = \sqrt{\hbar\mathbb{G}/c^5}$ for $\overset{(1\&3)}{\mapsto}$ Planck Scale Wholly Derived System of Units (þ – thorn)

Invoking the (3) transformation with Z_0, k, $\hbar$, c and $\mathbb{G}$ yields the low-precision rationalized Planck units

$$\begin{array}{rcl}
3.767 \times 10^{2}\ {}^{\mathrm{J{\cdot}s}}/_{\mathrm{C}^2} &= Z_0 =& 1\ {}^{\text{þe·þt}}/_{\text{þq}^2}\\
1.381 \times 10^{-23}\ {}^{\mathrm{J}}/_{\mathrm{K}} &= k =& 1\ {}^{\text{þe}}/_{\text{þa}}\\
1.055 \times 10^{-34}\ \mathrm{J{\cdot}s} &= \hbar =& 1\ \text{þe·þt}\\
2.998 \times 10^{8}\ {}^{\mathrm{m}}/_{\mathrm{s}} &= c =& 1\ {}^{\text{þl}}/_{\text{þt}}\\
8.387 \times 10^{-10}\ {}^{\mathrm{N{\cdot}m^2}}/_{\mathrm{kg}^2} &= \mathbb{G} =& 1\ {}^{\text{þf·þl}^2}/_{\text{þm}^2}
\end{array}
\quad\overset{(3)}{\Rightarrow}\quad
\begin{array}{lcl}
1\ \text{þq} &=& 5.291 \times 10^{-19}\ \mathrm{C}\\
1\ \text{þa} &=& 3.997 \times 10^{31}\ \mathrm{K}\\
1\ \text{þm} &=& 6.140 \times 10^{-9}\ \mathrm{kg}\\
1\ \text{þl} &=& 5.729 \times 10^{-35}\ \mathrm{m}\\
1\ \text{þt} &=& 1.911 \times 10^{-43}\ \mathrm{s}
\end{array}$$

These units are theoretically interesting but impractical. 1 þq, the unit of charge, is the only basis unit independent of $\mathbb{G}$ and the only basis unit with usable precision (which we have not displayed here).

Overview of $T \approx \sqrt{\hbar\mathbb{G}/c^5}$ for $\overset{(1)}{\mapsto}$ Planck Scale Time Derived Units (þſ, þd, þb, þß, þdd and þbb)

Let the bullet placeholder (●) represent a radix symbol ſ, d or b in a unit system (þ● or þ●●). Invoking the (1) transformation with Z_0, k, $\hbar$, c and the parameter T equal to 1 þ●t or 1 þ●●t which are defined exactly in terms of a second or a day will yield units for length, mass and temperature with exact conversions to SI. The derived unit of charge is independent of T so that $1\ \text{þ●●q} = 1\ \text{þ●q} = 1\ \text{þq}$ already has usable precision.

Consider the Planck scale denary units (þd); we can approximate 1 þt by 1 þdt so that

- 1 þdt will be a simplest-possible decimal fraction of a second
- $\frac{1\,\text{þdt}}{1\,\text{s}}$ has the reciprocal $\frac{1\,\text{s}}{1\,\text{þdt}}$ which will be a large power of ten
- $\frac{1\,\text{þdt}}{1\,\text{s}}$ when expressed in scientific notation will have a significand of one
- 1 þdt will be 1.0×10^{x} s where x is an integer close to $\log_{10}(\frac{1\,\text{þt}}{1\,\text{s}})$
- $\log_{10}(\frac{1\,\text{þdt}}{1\,\text{s}})$ is an integer which will be close to $\log_{10}(\frac{1\,\text{þt}}{1\,\text{s}})$

Generally, we choose

$$\begin{array}{llccc}
 & & \log_{60} & \log_{10} & \log_{2} \\
\hline
\star\star & {}^{1\,\text{þt}}/_{1\,\text{s}} & \boxed{-24.02} & \boxed{-42.72} & \boxed{-141.91}\\
\star\star\star & {}^{1\,\text{þt}}/_{1\,\text{day}} & -26.80 & -47.66 & -158.31
\end{array}
\quad\Longrightarrow\quad
\begin{array}{rccccr}
 & \text{ſ} & \text{d} & \text{b} & & \\
\hline
1\,\text{þ•t} = & \boxed{60^{-24}} & \boxed{10^{-43}} & \boxed{2^{-142}} & \text{s} & \star\star\\
1\,\text{þ••t} = & 60^{-27} & 10^{-48} & 2^{-158} & \text{day} & \star\star\star
\end{array}$$

★★ T as fractions of 1 s $\approx \sqrt{\hbar\mathbb{G}/c^5}$ for $\overset{(1)}{\mapsto}$ Planck Scale Time Derived Units (þſ, þd and þb)

Invoking the (1) transformation with Z_0, k, $\hbar$, c and $T = 1$ þ•t (where the bullet ranges over ſ, d or b) yields exact Planck scale units (þſ, þd or þb); we note that the unit of charge is always the same and omit it; and we only display limited precision for brevity

1 þſt = 60^{-24} s for Units (þſ)

$$\begin{array}{lcl}
1\ \text{þſa} &=& 3.619 \times 10^{31}\ \mathrm{K}\\
1\ \text{þſm} &=& 5.560 \times 10^{-9}\ \mathrm{kg}\\
1\ \text{þſl} &=& 6.327 \times 10^{-35}\ \mathrm{m}\\
1\ \text{þſt} &=& 2.110 \times 10^{-43}\ \mathrm{s}
\end{array}$$

$$\frac{\mathbb{G}}{1\ {}^{\text{þſf·þſl}^2}/_{\text{þſm}^2}} \overset{(2)}{=} \left(\frac{1\,\text{þſt}}{1\,\text{þt}}\right)^{-2} = 0.820\,1$$

1 þdt = 10^{-43} s for Units (þd)

$$\begin{array}{lcl}
1\ \text{þda} &=& 7.638 \times 10^{31}\ \mathrm{K}\\
1\ \text{þdm} &=& 1.173 \times 10^{-8}\ \mathrm{kg}\\
1\ \text{þdl} &=& 2.998 \times 10^{-35}\ \mathrm{m}\\
1\ \text{þdt} &=& 1.000 \times 10^{-43}\ \mathrm{s}
\end{array}$$

$$\frac{\mathbb{G}}{1\ {}^{\text{þdf·þdl}^2}/_{\text{þdm}^2}} \overset{(2)}{=} \left(\frac{1\,\text{þdt}}{1\,\text{þt}}\right)^{-2} = 3.652$$

1 þbt = 2^{-142} s for Units (þb)

$$\begin{array}{lcl}
1\ \text{þba} &=& 4.258 \times 10^{31}\ \mathrm{K}\\
1\ \text{þbm} &=& 6.542 \times 10^{-9}\ \mathrm{kg}\\
1\ \text{þbl} &=& 5.377 \times 10^{-35}\ \mathrm{m}\\
1\ \text{þbt} &=& 1.794 \times 10^{-43}\ \mathrm{s}
\end{array}$$

$$\frac{\mathbb{G}}{1\ {}^{\text{þbf·þbl}^2}/_{\text{þbm}^2}} \overset{(2)}{=} \left(\frac{1\,\text{þbt}}{1\,\text{þt}}\right)^{-2} = 1.135$$

★★★ T as fractions of 1 day ≈ $\sqrt{\hbar\mathbb{G}/c^5}$ for Planck Scale Time Derived Units (þß, þdd and þbb)

Invoking the (1) transformation with Z_0, k, $\hbar$, c and $T = 1\,\text{þ••t}$ (where the bullet ranges as need for the system þß, þdd or þbb) yields more Planck scale units. To determine the best choice for the 1 þ••t we will refer to a logarithm identity

$$\log\left(\tfrac{1\,\text{þt}}{1\,\text{day}}\right) = \log\left(\tfrac{1\,\text{þt}}{1\,\text{s}}\right) + \log\left(\tfrac{1\,\text{s}}{1\,\text{day}}\right)$$

For sexagenary, denary and binary the rounded logs also add up as integers but, for logarithms to other bases, we may need to fudge when rounding to a close whole number to retain additivity. It is probably more important to honour the logs on the right of the identity so they should be rounded first to determine the most suitable integer on the left.

As before, we take omit of the unvarying unit of charge and we only display limited precision for brevity.

1 þßt = 60^{-27} day for (þß)

$$\begin{aligned}
1\,\text{þßa} &= 9.048 \times 10^{31}\,\text{K}\\
1\,\text{þßm} &= 1.390 \times 10^{-8}\,\text{kg}\\
1\,\text{þßl} &= 2.531 \times 10^{-35}\,\text{m}\\
1\,\text{þßt} &= 8.442 \times 10^{-44}\,\text{s}
\end{aligned}$$

$$\frac{\mathbb{G}}{1\,\text{þßf}{\cdot}\text{þßl}^2/\text{þßm}^2} \overset{(2)}{=} \left(\frac{1\,\text{þßt}}{1\,\text{þt}}\right)^{-2} = 5.125$$

1 þddt = 10^{-48} day for (þdd)

$$\begin{aligned}
1\,\text{þdda} &= 8.841 \times 10^{31}\,\text{K}\\
1\,\text{þddm} &= 1.358 \times 10^{-8}\,\text{kg}\\
1\,\text{þddl} &= 2.590 \times 10^{-35}\,\text{m}\\
1\,\text{þddt} &= 8.640 \times 10^{-44}\,\text{s}
\end{aligned}$$

$$\frac{\mathbb{G}}{1\,\text{þddf}{\cdot}\text{þddl}^2/\text{þddm}^2} \overset{(2)}{=} \left(\frac{1\,\text{þddt}}{1\,\text{þt}}\right)^{-2} = 4.893$$

1 þbbt = 2^{-158} day for (þbb)

$$\begin{aligned}
1\,\text{þbba} &= 3.230 \times 10^{31}\,\text{K}\\
1\,\text{þbbm} &= 4.962 \times 10^{-9}\,\text{kg}\\
1\,\text{þbbl} &= 7.089 \times 10^{-35}\,\text{m}\\
1\,\text{þbbt} &= 2.365 \times 10^{-43}\,\text{s}
\end{aligned}$$

$$\frac{\mathbb{G}}{1\,\text{þbbf}{\cdot}\text{þbbl}^2/\text{þbbm}^2} \overset{(2)}{=} \left(\frac{1\,\text{þbbt}}{1\,\text{þt}}\right)^{-2} = 0.653\,2$$

Common Scale Base Units As Shifted from Planck Scale Base Units

	$\log_{60}$	$\log_{10}$	$\log_{2}$
$^{1\,\text{mC}}/_{1\,\text{þq}}$	8.59	15.28	50.75
$^{1\,\text{K}}/_{1\,\text{þa}}$	-17.77	-31.60	-104.98
$^{1\,\text{hg}}/_{1\,\text{þm}}$	4.06	7.21	23.96
$^{1\,\text{dm}}/_{1\,\text{þl}}$	18.69	33.24	110.43
$^{1\,\text{s}}/_{1\,\text{þt}}$	24.02	42.72	141.91

$\Longrightarrow$

Shift for		ſ	d	b
charge [q]	F	60^9	10^{16}	2^{53}
temp. [a]	°	60^{-18}	10^{-32}	2^{-106}
mass [m]	q	60^4	10^7	2^{24}
length [l]	f	60^{19}	10^{34}	2^{112}
time [t]	h	60^{24}	10^{43}	2^{142}
				þ

Shifted (þ, þſ and þß) Units Realizing (ſ, ß and ſß) Units

$$\begin{aligned}
1\,\text{Fſq} &= 60^{9}\,\text{þq} = 5.332\,\text{mC}\\[1ex]
1\,{}^\circ\text{ſa} &= 60^{-18}\,\text{þa} = 0.393\,5\,\text{K}\\
1\,\text{qſm} &= 60^{4}\,\text{þm} = 79.57\,\text{g}\\
1\,\text{fſl} &= 60^{19}\,\text{þl} = 0.349\,1\,\text{m}\\
1\,\text{hſt} &= 60^{24}\,\text{þt} = 0.905\,6\,\text{s}\\[1ex]
1\,{}^\circ\text{ßa} &= 60^{-18}\,\text{þſa} = 0.356\,4\,\text{K}\\
1\,\text{qßm} &= 60^{4}\,\text{þſm} = 72.06\,\text{g}\\
1\,\text{fßl} &= 60^{19}\,\text{þſl} = 0.385\,5\,\text{m}\\
1\,\text{hßt} &= 60^{24}\,\text{þſt} = 1\,\text{s}\\[1ex]
1\,{}^\circ\text{ſßa} &= 60^{-18}\,\text{þßa} = 0.890\,9\,\text{K}\\
1\,\text{qſßm} &= 60^{4}\,\text{þßm} = 180.1\,\text{g}\\
1\,\text{fſßl} &= 60^{19}\,\text{þßl} = 0.154\,2\,\text{m}\\
1\,\text{hſßt} &= 60^{24}\,\text{þßt} = 0.400\,0\,\text{s}
\end{aligned}$$

Shifted (þ, þd and þdd) Units Realizing (d, dd and ddd) Units

$$\begin{aligned}
1\,\text{Fdq} &= 10^{16}\,\text{þq} = 5.291\,\text{mC}\\[1ex]
1\,{}^\circ\text{da} &= 10^{-32}\,\text{þa} = 0.399\,7\,\text{K}\\
1\,\text{qdm} &= 10^{7}\,\text{þm} = 61.40\,\text{g}\\
1\,\text{fdl} &= 10^{34}\,\text{þl} = 0.572\,9\,\text{m}\\
1\,\text{hdt} &= 10^{43}\,\text{þt} = 1.911\,\text{s}\\[1ex]
1\,{}^\circ\text{dda} &= 10^{-32}\,\text{þda} = 0.763\,8\,\text{K}\\
1\,\text{qddm} &= 10^{7}\,\text{þdm} = 117.3\,\text{g}\\
1\,\text{fddl} &= 10^{34}\,\text{þdl} = 0.299\,8\,\text{m}\\
1\,\text{hddt} &= 10^{43}\,\text{þdt} = 1\,\text{s}\\[1ex]
1\,{}^\circ\text{ddda} &= 10^{-32}\,\text{þdda} = 0.884\,1\,\text{K}\\
1\,\text{qdddm} &= 10^{7}\,\text{þddm} = 135.8\,\text{g}\\
1\,\text{fdddl} &= 10^{34}\,\text{þddl} = 0.259\,0\,\text{m}\\
1\,\text{hdddt} &= 10^{43}\,\text{þddt} = 0.864\,0\,\text{s}
\end{aligned}$$

Shifted (þ, þb and þbb) Units Realizing (b, bb and bbb) Units

$$\begin{aligned}
1\,\text{Fbq} &= 2^{53}\,\text{þq} = 4.766\,\text{mC}\\[1ex]
1\,{}^\circ\text{ba} &= 2^{-106}\,\text{þa} = 0.492\,6\,\text{K}\\
1\,\text{qbm} &= 2^{24}\,\text{þm} = 103.0\,\text{g}\\
1\,\text{fbl} &= 2^{112}\,\text{þl} = 0.297\,5\,\text{m}\\
1\,\text{hbt} &= 2^{142}\,\text{þt} = 1.065\,5\,\text{s}\\[1ex]
1\,{}^\circ\text{bba} &= 2^{-106}\,\text{þba} = 0.524\,9\,\text{K}\\
1\,\text{qbbm} &= 2^{24}\,\text{þbm} = 109.8\,\text{g}\\
1\,\text{fbbl} &= 2^{112}\,\text{þbl} = 0.279\,2\,\text{m}\\
1\,\text{hbbt} &= 2^{142}\,\text{þbt} = 1\,\text{s}\\[1ex]
1\,{}^\circ\text{bbba} &= 2^{-106}\,\text{þbba} = 0.398\,1\,\text{K}\\
1\,\text{qbbbm} &= 2^{24}\,\text{þbbm} = 83.25\,\text{g}\\
1\,\text{fbbbl} &= 2^{112}\,\text{þbbl} = 0.368\,1\,\text{m}\\
1\,\text{hbbbt} &= 2^{142}\,\text{þbbt} = 1.318\,\text{s}
\end{aligned}$$

Shifts are case agnostic (★, ★★ or ★★★) so all cases are shifted alike. The charge unit is case invariant and not repeated. Prefixed SI units are used for charge and mass to compress the display.

A Pretty Shift Table?

The reading and interpretation of the shift table above is obvious in context when all the base units are shown shifted by it. But it's not an ideal table; it's not self-documenting and doesn't suggest any generalized notation. It is only the names implied for the shifted units, implied by the common scale part of the new unit symbols that suggest we are creating whole new systems of coherent common scale units rather than just adding some secondary names to the existing Planck scale systems.

Common Scale Coherent Systems as Shifted from Planck Scale Coherent Systems

With coherence, shifting the five base units also shifts the myriad derived units realizing a whole new common scale system of units. Wrangling derived units is easily done without any sophisticated algebra, but matrix multiplication allows for a nice visualization. We can think of each shift from the table above as a conversion of a single common scale base unit to its representation in the Planck scale units of matching dimensionality. We'll also need the opposite conversion which is simply a matter of negating the exponents. To express this compactly we'll come up with a placeholder notation for the systems on either side of the shift/conversion. Note that there are nine (three radices of three cases each) system pairs.

Placeholder Notation for Systems Shift

case	(⊛⟼∗)		
★	(þ⟼ſ)	(þ⟼d)	(þ⟼b)
★★	(þſ⟼ß)	(þd⟼dd)	(þb⟼bb)
★★★	(þß⟼ſß)	(þdd⟼ddd)	(þbb⟼bbb)

$$
\left.\begin{array}{llllll}
\multicolumn{6}{c}{\text{Shift Conversions}}\\
\hline
1\,\mathrm{F}{\ast}\mathrm{q} & = & 60^{9} & 10^{16} & 2^{53} & \circledast\mathrm{q}\\
1\,{}^{\circ}{\ast}\mathrm{a} & = & 60^{-18} & 10^{-32} & 2^{-106} & \circledast\mathrm{a}\\
1\,\mathrm{q}{\ast}\mathrm{m} & = & 60^{4} & 10^{7} & 2^{24} & \circledast\mathrm{m}\\
1\,\mathrm{f}{\ast}\mathrm{l} & = & 60^{19} & 10^{34} & 2^{112} & \circledast\mathrm{l}\\
1\,\mathrm{h}{\ast}\mathrm{t} & = & 60^{24} & 10^{43} & 2^{142} & \circledast\mathrm{t}\\
\hline
\end{array}\right\}
\Leftrightarrow
\left\{\begin{array}{llllll}
\multicolumn{6}{c}{\text{Opposing Conversions}}\\
\hline
1\,\circledast\mathrm{q} & = & 60^{-9} & 10^{-16} & 2^{-53} & \mathrm{F}{\ast}\mathrm{q}\\
1\,\circledast\mathrm{a} & = & 60^{18} & 10^{32} & 2^{106} & {}^{\circ}{\ast}\mathrm{a}\\
1\,\circledast\mathrm{m} & = & 60^{-4} & 10^{-7} & 2^{-24} & \mathrm{q}{\ast}\mathrm{m}\\
1\,\circledast\mathrm{l} & = & 60^{-19} & 10^{-34} & 2^{-112} & \mathrm{f}{\ast}\mathrm{l}\\
1\,\circledast\mathrm{t} & = & 60^{-24} & 10^{-43} & 2^{-142} & \mathrm{h}{\ast}\mathrm{t}\\
\hline
\end{array}\right.
$$

We can directly express coherent derived unit conversion factors by multiplying dimensional row vectors by, for the given radix, the column vector of shift exponents. The bundled rows of dimensional row vectors can be matrix multiplied with the bundled columns for different radices allowing for a succinct expression. For example, for some purely mechanical units,

$$
\left.\begin{array}{r}
\text{impulse unit}\\
\text{force unit}\\
\hline
\text{work unit}\\
\text{action unit}
\end{array}\right\}
\left[\begin{array}{ccccc}
\mathsf{Q} & \Theta & \mathsf{M} & \mathsf{L} & \mathsf{T}\\
0 & 0 & 1 & 1 & -1\\
0 & 0 & 1 & 1 & -2\\
\hline
0 & 0 & 1 & 2 & -2\\
0 & 0 & 1 & 2 & -1
\end{array}\right]
\left[\begin{array}{c|c|c}
\text{ſ} & \text{d} & \text{b}\\
9 & 16 & 53\\
-18 & -32 & -106\\
4 & 7 & 24\\
19 & 34 & 112\\
24 & 43 & 142
\end{array}\right]
=
\left[\begin{array}{c|c|c}
\text{ſ} & \text{d} & \text{b}\\
-1 & -2 & -6\\
-25 & -45 & -148\\
\hline
-6 & -11 & -36\\
18 & 32 & 106
\end{array}\right]
$$

This matrix multiplication yields the exponents needed to represent the desired coherent common scale units in powers of the Planck scale units and, when negated, vice-versa

$$
\left.\begin{array}{llllll}
\hline
1\,\mathrm{dyn}{\ast}\mathrm{f}\cdot\mathrm{h}{\ast}\mathrm{t} & = & 60^{-1} & 10^{-2} & 2^{-6} & \circledast\mathrm{f}\cdot\circledast\mathrm{t}\\
1\,\mathrm{dyn}{\ast}\mathrm{f} & = & 60^{-25} & 10^{-45} & 2^{-148} & \circledast\mathrm{f}\\
\hline
1\,\mathrm{erg}{\ast}\mathrm{e} & = & 60^{-6} & 10^{-11} & 2^{-36} & \circledast\mathrm{e}\\
1\,\mathrm{erg}{\ast}\mathrm{e}\cdot\mathrm{h}{\ast}\mathrm{t} & = & 60^{18} & 10^{32} & 2^{106} & \circledast\mathrm{e}\cdot\circledast\mathrm{t}\\
\hline
\end{array}\right\}
\Leftrightarrow
\left\{\begin{array}{llllll}
\hline
1\,\circledast\mathrm{f}\cdot\circledast\mathrm{t} & = & 60^{1} & 10^{2} & 2^{6} & \mathrm{dyn}{\ast}\mathrm{f}\cdot\mathrm{h}{\ast}\mathrm{t}\\
1\,\circledast\mathrm{f} & = & 60^{25} & 10^{45} & 2^{148} & \mathrm{dyn}{\ast}\mathrm{f}\\
\hline
1\,\circledast\mathrm{e} & = & 60^{6} & 10^{11} & 2^{6} & \mathrm{erg}{\ast}\mathrm{e}\\
1\,\circledast\mathrm{e}\cdot\circledast\mathrm{t} & = & 60^{-18} & 10^{-32} & 2^{-106} & \mathrm{erg}{\ast}\mathrm{e}\cdot\mathrm{h}{\ast}\mathrm{t}\\
\hline
\end{array}\right.
$$

The opposing conversion is needed whenever a measurement is given in the Planck scale units and it needs to be expressed in the coherent common scale units. We substitute in the Planck scale units as expressed in common scale units and expand in order to expunge the Planck scale units entirely. A measurement is *invariant* with changes in base units, the change in the base units themselves is called *covariant* and the change in the expressed value of the measurement when expressed in the covariantly changed base units is called *contravariant*, the covariance and contravariance cancelling each other out to ensure the abstract but needful invariant.

In particular, for expressing the fundamental constants $Z_0 = 1\,{}^{\circledast\mathrm{e}\cdot\circledast\mathrm{t}}/_{\circledast\mathrm{q}^2}$, $k = 1\,{}^{\circledast\mathrm{e}}/_{\circledast\mathrm{a}}$, $\hbar = 1\,\circledast\mathrm{e}\cdot\circledast\mathrm{t}$, $c = 1\,{}^{\circledast\mathrm{l}}/_{\circledast\mathrm{t}}$ and, somewhat, $\mathbb{G} \approx 1\,{}^{\circledast\mathrm{f}\cdot\circledast\mathrm{l}^2}/_{\circledast\mathrm{m}^2}$ in a common scale system we need to do a coherent conversion from the corresponding Planck scale system. We can do this clunkily or elegantly. The clunky solution is to substitute in already known opposing conversions algebraically and build up solutions case-by-case

$$\text{e.g. for denary units (d, dd or ddd)} \qquad k = 1\,{}^{\circledast\mathrm{e}}/_{\circledast\mathrm{a}} = \frac{10^{11}\ \mathrm{erg{*}e}}{10^{32}\ {}^{\circ}{*}\mathrm{a}} = 10^{-21}\ {}^{\mathrm{erg{*}e}}/_{{}^{\circ}{*}\mathrm{a}}$$

The elegant solution is to use matrices to collate all the cases together. Multiplying the column vector of negated exponents on the left by the defining five-by-five matrix of the (3) transformation (the one doubling as a dimensionality annotation for the column of fundamental constants) yields the exponents needed to represent the appropriate Planck scale units in powers of the common scale units. Bundling the columns for different radices together allows these all to be expressed in a negation and a single matrix multiplication

$$-\left[\begin{array}{cc|ccc} \mathsf{Q} & \Theta & \mathsf{M} & \mathsf{L} & \mathsf{T} \\ -2 & 0 & 1 & 2 & -1 \\ 0 & -1 & 1 & 2 & -2 \\ \hline & & 1 & 2 & -1 \\ \multicolumn{2}{c|}{0} & 0 & 1 & -1 \\ & & -1 & 3 & -2 \end{array}\right] \left[\begin{array}{c|c|c} \mathrm{f} & \mathrm{d} & \mathrm{b} \\ 9 & 16 & 53 \\ -18 & -32 & -106 \\ 4 & 7 & 24 \\ 19 & 34 & 112 \\ 24 & 43 & 142 \end{array}\right] = \left[\begin{array}{c|c|c} \mathrm{f} & \mathrm{d} & \mathrm{b} \\ 0 & 0 & 0 \\ -12 & -21 & -70 \\ -18 & -32 & -106 \\ 5 & 9 & 30 \\ -5 & -9 & -28 \end{array}\right]$$

$$\Rightarrow \left\{\begin{array}{rlcllll} \hline Z_0 & = 1\,{}^{\circledast\mathrm{e}\cdot\circledast\mathrm{t}}/_{\circledast\mathrm{q}^2} & = & 1 & 1 & 1 & {}^{\mathrm{erg{*}e\cdot h{*}t}}/_{\mathrm{F{*}q}^2} \\ k & = 1\,{}^{\circledast\mathrm{e}}/_{\circledast\mathrm{a}} & = & 60^{-12} & 10^{-21} & 2^{-70} & {}^{\mathrm{erg{*}e}}/_{{}^{\circ}{*}\mathrm{a}} \\ \hbar & = 1\,\circledast\mathrm{e}\cdot\circledast\mathrm{t} & = & 60^{-18} & 10^{-32} & 2^{-106} & \mathrm{erg{*}e\cdot h{*}t} \\ c & = 1\,{}^{\circledast\mathrm{l}}/_{\circledast\mathrm{t}} & = & 60^{5} & 10^{9} & 2^{30} & {}^{\mathrm{f{*}l}}/_{\mathrm{h{*}t}} \\ \mathbb{G} & \approx 1\,{}^{\circledast\mathrm{f}\cdot\circledast\mathrm{l}^2}/_{\circledast\mathrm{m}^2} & = & 60^{-5} & 10^{-9} & 2^{-28} & {}^{\mathrm{dyn{*}f\cdot f{*}l}^2}/_{\mathrm{q{*}m}^2} \\ \hline \end{array}\right.$$

Fundamental Constants in Systems Shifted from the Planck Scale To reiterate, the fundamental constants of nature Z_0, k, $\hbar$, c and, for the wholly derived units, $\mathbb{G}$ will be represented by whole powers of the respective number base for these sexagenary (f), denary (d) and binary (b) shifted units. When T only approximates $\sqrt{{}^{\hbar\mathbb{G}}/_{c^5}}$ then, in the scientific notation for its own number base, $\mathbb{G}$ will have a small significand as determined by equation (2).

$$\left.\begin{array}{lcllll} \hline 1\,\mathrm{F{*}q} & = & 60^{9} & 10^{16} & 2^{53} & \circledast\mathrm{q} \\ 1\,{}^{\circ}{*}\mathrm{a} & = & 60^{-18} & 10^{-32} & 2^{-106} & \circledast\mathrm{a} \\ 1\,\mathrm{q{*}m} & = & 60^{4} & 10^{7} & 2^{24} & \circledast\mathrm{m} \\ 1\,\mathrm{f{*}l} & = & 60^{19} & 10^{34} & 2^{112} & \circledast\mathrm{l} \\ 1\,\mathrm{h{*}t} & = & 60^{24} & 10^{43} & 2^{142} & \circledast\mathrm{t} \\ \hline \end{array}\right\} \Leftrightarrow \left\{\begin{array}{rcllll} \hline Z_0\,[\mathsf{Q}^{-2}\mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-1}] & = & 1 & 1 & 1 & {}^{\mathrm{erg{*}e\cdot h{*}t}}/_{\mathrm{F{*}q}^2} \\ k\,[\Theta^{-1}\mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-2}] & = & 60^{-12} & 10^{-21} & 2^{-70} & {}^{\mathrm{erg{*}e}}/_{{}^{\circ}{*}\mathrm{a}} \\ \hbar\,[\mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-1}] & = & 60^{-18} & 10^{-32} & 2^{-106} & \mathrm{erg{*}e\cdot h{*}t} \\ c\,[\mathsf{L}\mathsf{T}^{-1}] & = & 60^{5} & 10^{9} & 2^{30} & {}^{\mathrm{f{*}l}}/_{\mathrm{h{*}t}} \\ \mathbb{G}\,[\mathsf{M}^{-1}\mathsf{L}^{3}\mathsf{T}^{-2}] & \approx & 60^{-5} & 10^{-9} & 2^{-28} & {}^{\mathrm{dyn{*}f\cdot f{*}l}^2}/_{\mathrm{q{*}m}^2} \\ \hline \end{array}\right.$$

We have $\mathbb{G} = 10^{-9}\ {}^{\mathrm{dyndf\cdot fdl}^2}/_{\mathrm{qdm}^2}$ exactly in the wholly derived denary units; while, in the subtly different time derived units, we have $\mathbb{G} = 3.652 \times 10^{-9}\ {}^{\mathrm{dynddf\cdot fddl}^2}/_{\mathrm{qddm}^2}$ and $\mathbb{G} = 4.893 \times 10^{-9}\ {}^{\mathrm{dyndddf\cdot fdddl}^2}/_{\mathrm{qdddm}^2}$.

Practically-Sized Constants Leading to Practically-Sized Units

The constant of nature Z_0 would already derive a practically sized unit. Dividing other constants k, $\hbar = {}^{h}\!/_{2\pi}$, c and, for a wholly derived system, $\mathbb{G} = 4\pi G$ by powers of 60, 10 or 2 before calculating the wholly derived (via (3)) or the $T{=}1$ s derived (via (1)) base units directly yields sexagenary, denary and binary units at a convenient size with no further scaling needed; Dividing $T{=}1$ day into parts via a power of the base, rather than the using the second directly, better honours the numbering system; the divided day yields time derived (via (1)) base units in systems where hours, minutes and seconds make little sense. For example, the 24 hour day can be divided in 10×100×100 parts yielding a conveniently-sized unit of time that is a little bit shorter than a second.

For example, dividing c by 10^9 before the derivation yields units T and L such that $c = 1{\times}10^9\ \mathrm{L{\cdot}T^{-1}}$.

Weirder powers of 20, 12 or 3 would yield vicenary (ϕ), duodenary (ȸ) and ternary (θ) systems.

foundational constants							divisors
Z_0	÷	1	1	1	1	1	1
k	÷	60^{-12}	10^{-21}	2^{-70}	20^{-16}	12^{-20}	3^{-44}
$\hbar$	÷	60^{-18}	10^{-32}	2^{-106}	20^{-24}	12^{-30}	3^{-67}
c	÷	60^{5}	10^{9}	2^{30}	20^{7}	12^{8}	3^{19}
$\mathbb{G}$	÷	60^{-5}	10^{-9}	2^{-28}	20^{-7}	12^{-9}	3^{-17}
1 s	÷	1	1	1	1	1	1
1 day	÷	60^{3}	10^{5}	2^{16}	20^{4}	12^{5}	3^{10}

e.g. interpreting the table for wholly derived denary units would result in the system where

$$Z_0 = 1\ {}^{\mathrm{ergde{\cdot}hdt}}\!/_{\mathrm{Fdq^2}}$$
$$k = 1{\times}10^{-21}\ {}^{\mathrm{ergde}}\!/_{\circ\mathrm{da}}$$
$$\hbar = 1{\times}10^{-32}\ \mathrm{ergde{\cdot}hdt}$$
$$c = 1{\times}10^{9}\ {}^{\mathrm{fdl}}\!/_{\mathrm{hdt}}$$
$$\mathbb{G} = 1{\times}10^{-9}\ {}^{\mathrm{dyndf{\cdot}fdl^2}}\!/_{\mathrm{qdm^2}}$$

Parity and the Ratio between the Practically-Sized and Planck Scale Units

A parity relation on the number base logarithm of the divisor for $\hbar$ determines whether the conveniently sized charge unit derived this way is rationally commensurable with the Planck scale charge unit; an even logarithm leads to rationally commensurable charge units, an odd logarithm does not and introduces a factor of the square root of the number base.

A parity relation on the sum of logarithms of the divisors for $\hbar$, c and $\mathbb{G}$ determines whether the wholly determined and conveniently sized units of time, distance, mass and temperature are rationally commensurable with their respective Planck scale units; an even sum of logarithms leads to rationally commensurable units.

Having our practically-sized system of units be rationally commensurable with Planck scale units seems desirable (and given its derivation is straightforward) but would only rarely be a useful.

The divisors for the number base 60, 10 and base 2 case come from their scaled cousins described earlier in this document and so automatically satisfy the parity relation; in this case, the scaled Planck units are necessarily the same as the units directly derived from the divided constants. The vicenary (number base 20) case is also well behaved and could have defined by a shift from the Planck scale.

The number base 12 and base 3 units work better when the divided constants result in an odd parity relation and the practically derived units are thus rationally incommensurable with the Planck units. The freedom to shift the derived duodenary units by $\sqrt{12}$ and the ternary units by $\sqrt{3}$ is more useful.

e.g. Denary Systems of Units (d, dd and ddd)

denary divisors	
$Z_0 \div 1$	$= 376.7\,\text{J·s}/\text{C}^2$
$k \div 10^{-21}$	$= 0.013\,81\,\text{J}/\text{K}$
$\hbar \div 10^{-32}$	$= 0.010\,55\,\text{J·s}$
$c \div 10^{9}$	$= 0.299\,8\,\text{m}/\text{s}$
$\mathbb{G} \div 10^{-9}$	$= 0.838\,7\,\text{N·m}^2/\text{kg}^2$
$1\,\text{s} \div 1$	$= 1\,\text{s}$
$1\,\text{day} \div 10^{5}$	$= 0.864\,0\,\text{s}$

time	length	mass	force	work	temp.	charge
hdt	fdl	qdm	dyndf	ergde	°da	Fdq
hddt	fddl	qddm	dynddf	ergdde	°dda	Fddq
hdddt	fdddl	qdddm	dyndddf	ergddde	°ddda	Fdddq

T parameter for the (1) transformation

$$1\,\text{hdt} = \sqrt{\frac{(0.010\,55\,\text{J·s})\cdot(0.838\,7\,\text{N·m}^2/\text{kg}^2)}{(0.299\,8\,\text{m}/\text{s})^5}} = 1.911\,\text{s}$$

$$1\,\text{hddt} = 1\,\text{s}$$

$$1\,\text{hdddt} = 10^{-5}\,\text{day} = \frac{24\times60\times60\,\text{s}}{10\times100\times100} = \frac{86\,400\,\text{s}}{100\,000} = 0.864\,0\,\text{s}$$

Note that feeding the parameter 1 hdt into the (1) transformation is the same as running the (3) transformation. Note also that $\frac{1\,\text{hdddt}}{1\,\text{hddt}} = \frac{10^{-5}\,\text{day}}{1\,\text{s}} = \frac{1\,\text{þddt}}{1\,\text{þdt}}$ ensuring consistency with the previous derivation of the ddd units as being shifted from the Planck scale. We express inputs and outputs to the transformations in fully coherent units (without SI prefixed forms) for clarity in numerical calculation.

$$\begin{array}{rlll}
Z_0 \div 1 & = 376.7\,\text{J·s}/\text{C}^2 & = 1\,\text{ergde·hdt}/\text{Fdq}^2 & & 1\,\text{Fdq} & = 0.005\,291\,\text{C} \\
k \div 10^{-21} & = 0.013\,81\,\text{J}/\text{K} & = 1\,\text{ergde}/{}^\circ\text{da} & & 1\,{}^\circ\text{da} & = 0.399\,7\,\text{K} \\
\hbar \div 10^{-32} & = 0.010\,55\,\text{J·s} & = 1\,\text{ergde·hdt} & \overset{(3)}{\Rightarrow} & 1\,\text{qdm} & = 0.061\,40\,\text{kg} \\
c \div 10^{9} & = 0.299\,8\,\text{m}/\text{s} & = 1\,\text{fdl}/\text{hdt} & & 1\,\text{fdl} & = 0.572\,9\,\text{m} \\
\mathbb{G} \div 10^{-9} & = 0.838\,7\,\text{N·m}^2/\text{kg}^2 & = 1\,\text{dyndf·fdl}^2/\text{qdm}^2 & & 1\,\text{hdt} & = 1.911\,\text{s}
\end{array}$$

$$\begin{array}{rlll}
Z_0 \div 1 & = 376.7\,\text{J·s}/\text{C}^2 & = 1\,\text{ergdde·hddt}/\text{Fddq}^2 & & 1\,\text{Fddq} & = 0.005\,291\,\text{C} \\
k \div 10^{-21} & = 0.013\,81\,\text{J}/\text{K} & = 1\,\text{ergdde}/{}^\circ\text{dda} & & 1\,{}^\circ\text{dda} & = 0.763\,8\,\text{K} \\
\hbar \div 10^{-32} & = 0.010\,55\,\text{J·s} & = 1\,\text{ergdde·hddt} & \overset{(1)}{\Rightarrow} & 1\,\text{qddm} & = 0.117\,3\,\text{kg} \\
c \div 10^{9} & = 0.299\,8\,\text{m}/\text{s} & = 1\,\text{fddl}/\text{hddt} & & 1\,\text{fddl} & = 0.299\,8\,\text{m} \\
1\,\text{s} \div 1 & = 1\,\text{s} & = 1\,\text{hddt} & & 1\,\text{hddt} & = 1\,\text{s}
\end{array}$$

$$\therefore\ \frac{\mathbb{G}\div10^{-9}}{1\,\text{dynddf·fddl}^2/\text{qddm}^2} \overset{(2)}{=} \left(\frac{1\,\text{hddt}}{1\,\text{hdt}}\right)^{-2} = \left(\frac{1\,\text{hdt}}{1\,\text{hddt}}\right)^{2} = 3.652$$

$$\begin{array}{rlll}
Z_0 \div 1 & = 376.7\,\text{J·s}/\text{C}^2 & = 1\,\text{ergddde·hdddt}/\text{Fdddq}^2 & & 1\,\text{Fdddq} & = 0.005\,291\,\text{C} \\
k \div 10^{-21} & = 0.013\,81\,\text{J}/\text{K} & = 1\,\text{ergddde}/{}^\circ\text{ddda} & & 1\,{}^\circ\text{ddda} & = 0.884\,1\,\text{K} \\
\hbar \div 10^{-32} & = 0.010\,55\,\text{J·s} & = 1\,\text{ergddde·hdddt} & \overset{(1)}{\Rightarrow} & 1\,\text{qdddm} & = 0.135\,8\,\text{kg} \\
c \div 10^{9} & = 0.299\,8\,\text{m}/\text{s} & = 1\,\text{fdddl}/\text{hdddt} & & 1\,\text{fdddl} & = 0.259\,0\,\text{m} \\
1\,\text{day} \div 10^{5} & = 0.864\,0\,\text{s} & = 1\,\text{hdddt} & & 1\,\text{hdddt} & = 0.864\,0\,\text{s}
\end{array}$$

$$\therefore\ \frac{\mathbb{G}\div10^{-9}}{1\,\text{dyndddf·fdddl}^2/\text{qdddm}^2} \overset{(2)}{=} \left(\frac{1\,\text{hdddt}}{1\,\text{hdt}}\right)^{-2} = \left(\frac{1\,\text{hdt}}{1\,\text{hdddt}}\right)^{2} = 4.893$$

We have repeated the charge units $1\,\text{Fdq} = 1\,\text{Fddq} = 1\,\text{Fdddq}$ which are independent of the time parameter in the (1) transformation; otherwise these are exactly the same values as we calculated by shifting from the Planck scale. We would do well to follow our former presentation by using prefixed SI forms for the output units (certainly the mC and possibly the g) for a more readable presentation of the numbers.

Comparing Three Ways of Deriving Practically Sized Units

Together with the wholly derived Planck scale system of units þ we will also define two systems of time derived units ð and ðð at no sensible scale. We will use these as intermediaries between our derivation from fundamental constants divided by powers of the number base and the shifted units we first investigated.

$T = 1\,\text{s}$ for Time Derived System of Units (ð – eth) at No Sensible Scale

$$\begin{array}{rcl c rcl}
Z_0 &=& 1\,{}^{\text{ðe·ðt}}/_{\text{ðq}^2} & & 1\,\text{ðq} &=& 5.291 \times 10^{-19}\,\text{C} \\
k &=& 1\,{}^{\text{ðe}}/_{\text{ða}} & & 1\,\text{ða} &=& 7.638 \times 10^{-12}\,\text{K} \\
\hbar &=& 1\,\text{ðe·ðt} & \overset{(1)}{\Rightarrow} & 1\,\text{ðm} &=& 1.173 \times 10^{-51}\,\text{kg} \\
c &=& 1\,{}^{\text{ðl}}/_{\text{ðt}} & & 1\,\text{ðl} &=& 2.998 \times 10^{8}\,\text{m} \\
1\,\text{s} &=& 1\,\text{ðt} & & 1\,\text{ðt} &=& 1\,\text{s}
\end{array}$$

$T = 1\,\text{day}$ for Time Derived System of Units (ðð) at No Sensible Scale

$$\begin{array}{rcl c rcl}
Z_0 &=& 1\,{}^{\text{ððe·ððt}}/_{\text{ððq}^2} & & 1\,\text{ððq} &=& 5.291 \times 10^{-19}\,\text{C} \\
k &=& 1\,{}^{\text{ððe}}/_{\text{ðða}} & & 1\,\text{ðða} &=& 8.841 \times 10^{-17}\,\text{K} \\
\hbar &=& 1\,\text{ððe·ððt} & \overset{(1)}{\Rightarrow} & 1\,\text{ððm} &=& 1.358 \times 10^{-56}\,\text{kg} \\
c &=& 1\,{}^{\text{ððl}}/_{\text{ððt}} & & 1\,\text{ððl} &=& 2.590 \times 10^{13}\,\text{m} \\
1\,\text{day} &=& 1\,\text{ððt} & & 1\,\text{ððt} &=& 8.640 \times 10^{4}\,\text{s}
\end{array}$$

Note that the ð and ðð system of units are rationally commensurable via a factor of $86400 = 24{\times}60{\times}60$. The length 1 ðl is a lightsecond and 1 ððl a lightday; only the time and length units merit having names. These two systems of units are of no great interest by themselves but do allow us the following reckoning. Starting with the desired form for the fundamental constants we can use: the (3) transformation to back derive to the already known values of þ system; and the (1) transformation to back derive to the already known values of the ð and ðð systems. This works in any number base but, taking as a specific example, the denary system d is shifted from the þ system, the denary system dd is shifted from the ð system and the denary system ddd is shifted from the ðð system.

$$\begin{array}{lcccl c rcl}
1\,{}^{\text{ergde·hdt}}/_{\text{Fdq}^2} &=& Z_0 &=& 1\,{}^{\text{þe·þt}}/_{\text{þq}^2} & & 1\,\text{þq} &=& 10^{-16}\,\text{Fdq} \\
10^{-21}\,{}^{\text{ergde}}/_{{}^\circ\text{da}} &=& k &=& 1\,{}^{\text{þe}}/_{\text{þa}} & & 1\,\text{þa} &=& 10^{32}\,{}^\circ\text{da} \\
10^{-32}\,\text{ergde·hdt} &=& \hbar &=& 1\,\text{þe·þt} & \overset{(3)}{\Rightarrow} & 1\,\text{þm} &=& 10^{-7}\,\text{qdm} \\
10^{9}\,{}^{\text{fdl}}/_{\text{hdt}} &=& c &=& 1\,{}^{\text{þl}}/_{\text{þt}} & & 1\,\text{þl} &=& 10^{-34}\,\text{fdl} \\
10^{-9}\,{}^{\text{dyndf·fdl}^2}/_{\text{qdm}^2} &=& \mathbb{G} &=& 1\,{}^{\text{þf·þl}^2}/_{\text{þm}^2} & & 1\,\text{þt} &=& 10^{-43}\,\text{hdt}
\end{array}$$

$$\begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & -1 & 1/2 & 5/2 & -1/2 \\ & & 1/2 & 1/2 & -1/2 \\ & 0 & 1/2 & -3/2 & 1/2 \\ & & 1/2 & -5/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ -9 \end{bmatrix} = \begin{bmatrix} -16 \\ 32 \\ -7 \\ -34 \\ -43 \end{bmatrix}$$

$$\begin{array}{lcccl c rcl}
1\,{}^{\text{ergdde·s}}/_{\text{Fddq}^2} &=& Z_0 &=& 1\,{}^{\text{ðe·ðt}}/_{\text{ðq}^2} & & 1\,\text{ðq} &=& 10^{-16}\,\text{Fddq} \\
10^{-21}\,{}^{\text{ergdde}}/_{{}^\circ\text{dda}} &=& k &=& 1\,{}^{\text{ðe}}/_{\text{ða}} & & 1\,\text{ða} &=& 10^{-11}\,{}^\circ\text{dda} \\
10^{-32}\,\text{ergdde·hddt} &=& \hbar &=& 1\,\text{ðe·ðt} & \overset{(1)}{\Rightarrow} & 1\,\text{ðm} &=& 10^{-50}\,\text{qddm} \\
10^{9}\,{}^{\text{fddl}}/_{\text{hddt}} &=& c &=& 1\,{}^{\text{ðl}}/_{\text{ðt}} & & 1\,\text{ðl} &=& 10^{9}\,\text{fddl} \\
1\,\text{hddt} &=& T &=& 1\,\text{ðt} & & 1\,\text{ðt} &=& 1\,\text{hddt}
\end{array}$$

$$\begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ & & 1 & -2 & -1 \\ & & & 1 & 1 \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} -16 \\ -11 \\ -50 \\ 9 \\ 0 \end{bmatrix}$$

$$\begin{array}{lcccl c rcl}
1\,{}^{\text{ergddde·s}}/_{\text{Fdddq}^2} &=& Z_0 &=& 1\,{}^{\text{ððe·s}}/_{\text{ððq}^2} & & 1\,\text{ððq} &=& 10^{-16}\,\text{Fdddq} \\
10^{-21}\,{}^{\text{ergddde}}/_{{}^\circ\text{ddda}} &=& k &=& 1\,{}^{\text{ððe}}/_{\text{ðða}} & & 1\,\text{ðða} &=& 10^{-16}\,{}^\circ\text{ddda} \\
10^{-32}\,\text{ergddde·hdddt} &=& \hbar &=& 1\,\text{ððe·ððt} & \overset{(1)}{\Rightarrow} & 1\,\text{ððm} &=& 10^{-55}\,\text{qdddm} \\
10^{9}\,{}^{\text{fdddl}}/_{\text{hdddt}} &=& c &=& 1\,{}^{\text{ððl}}/_{\text{ððt}} & & 1\,\text{ððl} &=& 10^{14}\,\text{fdddl} \\
10^{5}\,\text{hdddt} &=& T &=& 1\,\text{ððt} & & 1\,\text{ððt} &=& 10^{5}\,\text{hdddt}
\end{array}$$

$$\begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ & & 1 & -2 & -1 \\ & & & 1 & 1 \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} -16 \\ -16 \\ -55 \\ 14 \\ 5 \end{bmatrix}$$

So we simply divide the known values of the þ, ð and ðð by their respective derived powers to realize the units of the d, dd and ddd systems. This is more complicated that the direct derivation of the denary units from the numeric value of fundamental constants in the desired form but it does serve to show the hidden connection between systems tuned to different number bases.

This presentation also makes clear how half integral exponents can appear when the parity relation is odd; such shifts work perfectly well but lead to rationally incommensurable units by definition.

Our initial method of deriving the time derived denary units from base ten friendly approximations 1 þdt and 1 þddt of the Planck scale 1 þt simultaneously scales the systems þ → d, þd → dd and þdd → ddd. This needs to be consistent with the base ten friendly division of a day 1 hdddt as an approximation of $1\,\text{hddt} = 1\,\text{s}$. There is no associativity that assures the best approximations require $\frac{1\,\text{þddt}}{1\,\text{þdt}} = \frac{1\,\text{hdddt}}{1\,\text{hddt}}$ but as long as it does hold all our three different derivations for time derived units must agree with each other.

Chosen Coherence in Reference Units for a Nice Presentation

We have chosen to size our derived natural units with a unit of mass somewhat less than a kilogramme and a unit of charge much less than a coulomb. This makes it tricky to express the values of the conversions between systems in a numeric table which directly express the (1) and (3) transformations while avoiding scientific notation for those numbers. We can state the fundamental constants of nature not in SI, but in the coherent system of second-metre-gramme-kelvin-millicoulomb units by juggling a power of a thousand here or there; After applying our transformations we get a nicely scaled table of numbers which needs little or no formatting

Conversion from Coherent Second-Metre-Gramme-Kelvin-Millicoulomb Units

Z_0	0.376 730 313 412			$mC^{-2}\cdot g\cdot m^{2}\cdot s^{-1}$
	sexagenary ß, f	**denary d**	**binary b**	
k	$30.053\,723\,554\times60^{-12}$	$13.806\,490\,000\times10^{-21}$	$16.299\,826\,406\times2^{-70}$	$K^{-1}\cdot g\cdot m^{2}\cdot s^{-2}$
$\hbar$	$10.710\,226\,810\times60^{-18}$	$10.545\,718\,176\times10^{-32}$	$8.555\,703\,025\times2^{-106}$	$g\cdot m^{2}\cdot s^{-1}$
c	$0.385\,535\,568\times60^{5}$	$0.299\,792\,458\times10^{9}$	$0.279\,203\,484\times2^{30}$	$m\cdot s^{-1}$
$\mathbb{G}$	$0.000\,652\,187\times60^{-5}$	$0.000\,838\,717\times10^{-95}$	$0.000\,225\,141\times2^{-285}$	$g^{-1}\cdot m^{3}\cdot s^{-2}$
F●q	5.331 925 229	5.290 817 692	4.765 544 917	mC
°•a	0.393 528	0.399 667	0.492 628	K
q•m	79.569 318	61.396 079	103.005 528	g
f•l	0.349 131	0.572 947	0.297 491	m
h•t	0.905 574	1.911 147	1.065 500	s
°••a	0.356 369 379	0.763 823 258	0.524 895 346	K
q••m	72.055 936 326	117.336 939 202	109.752 402 408	g
f••l	0.385 535 568	0.299 792 458	0.279 203 484	m
°•••a	0.890 923 448	0.884 054 697	0.398 142 840	K
q•••m	180.139 840 815	135.806 642 595	83.249 229 678	g
f•••l	0.154 214 227	0.259 020 684	0.368 090 530	m
h•••t	0.400 000 000	0.864 000 000	1.318 359 375	s
	vicenary ϕ	**duodenary ȸ**	**ternary θ**	
k	$9.048\,221\,286\times20^{-16}$	$52.930\,768\,998\times12^{-20}$	$13.596\,229\,613\times3^{-44}$	$K^{-1}\cdot g\cdot m^{2}\cdot s^{-2}$
$\hbar$	$1.769\,277\,917\times20^{-24}$	$25.033\,037\,071\times12^{-30}$	$9.776\,878\,706\times3^{-67}$	$g\cdot m^{2}\cdot s^{-1}$
c	$0.234\,212\,858\times20^{7}$	$0.697\,221\,442\times12^{8}$	$0.257\,938\,912\times3^{19}$	$m\cdot s^{-1}$
$\mathbb{G}$	$0.001\,073\,558\times20^{-7}$	$0.004\,327\,597\times12^{-9}$	$0.000\,108\,312\times3^{-17}$	$g^{-1}\cdot m^{3}\cdot s^{-2}$
F●q	2.167 118 927	8.151 574 232	5.094 303 677	mC
°•a	0.119 110	0.583 246	0.746 681	K
q•m	19.646 745	63.506 581	152.587 896	g
f•l	0.384 498	0.565 358	0.248 406	m
h•t	1.641 662	0.810 873	0.963 044	s
°••a	0.195 538 754	0.472 939 229	0.719 087 496	K
q••m	32.253 332 255	51.495 831 304	146.948 954 583	g
f••l	0.234 212 858	0.697 221 442	0.257 938 912	m
°•••a	0.362 108 804	1.362 064 979	0.491 451 360	K
q•••m	59.728 393 065	148.307 994 155	100.430 426 148	g
f•••l	0.126 474 943	0.242 090 778	0.377 414 046	m
h•••t	0.540 000 000	0.347 222 222	1.463 191 587	s