

Deriving Natural Units in a Convenient Number Base

Derived vs. Fundamental Units in a System of Natural Units

In a natural system the difference between derived and fundamental units is a matter of degree; indeed, in a system of wholly derived units all measurements may be considered unitless.

basis	time	length	speed	mass	impulse	force	work	action	temp.	charge
TLMΘQ		L	LT ⁻¹	M	MLT ⁻¹	MLT ⁻²	ML ² T ⁻²	ML ² T ⁻¹	Θ	Q
TLM	T								ML ² T ⁻²	M ^{1/2} LT ^{-1/2}
TM		T	1		M	MT ⁻¹	M	MT	M	M ^{1/2} T ^{1/2}
T				T ⁻¹	T ⁻¹	T ⁻²	T ⁻¹	1	T ⁻¹	1

Within a term, units will be written in a reverse fashion with the more fundamental on the right and the more derived on the left. For our natural units we will set aside a few useful names and abbreviations based on a triple of (common name abbrev. / system of units abbrev. / dimensionality abbrev.)

time [t]	length [l]	mass [m]	force [f]	work [e]	temp. [a]	charge [q]
h•t	f•l	q•m	dyn•f	erg•e	°•a	F•q
heartbeat	neofoot	quatermass	neodyne	neoerg	neodegree	neofranklin

Time Derived System of Units (given an arbitrary fundamental time T)

$$\left. \begin{aligned} Z_0 &= 1 \text{ Q}^{-2} \cdot \text{M} \cdot \text{L}^2 \cdot \text{T}^{-1} & 1 \text{ Q} &= \sqrt{\hbar/Z_0} \\ k &= 1 \text{ } \Theta^{-1} \cdot \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} & 1 \text{ } \Theta &= \frac{\hbar}{kT} \\ \hbar &= 1 \text{ M} \cdot \text{L}^2 \cdot \text{T}^{-1} & \Leftrightarrow 1 \text{ M} &= \frac{\hbar}{c^2 T} \\ c &= 1 \text{ L} \cdot \text{T}^{-1} & 1 \text{ L} &= cT \\ T &= 1 \text{ T} & 1 \text{ T} &= T \end{aligned} \right\} \left(\begin{array}{ccc|ccc} -2 & 0 & 1 & 2 & -1 & \\ & -1 & 1 & 2 & -2 & \\ & & 1 & 2 & -1 & \\ 0 & & & 1 & -1 & \\ & & & & & 1 \end{array} \right)^{-1} = \left(\begin{array}{ccc|ccc} -1/2 & 0 & 1/2 & 0 & 0 & \\ & -1 & 1 & 0 & -1 & \\ & & 1 & -2 & -1 & \\ & 0 & & 1 & 1 & \\ & & & & & 1 \end{array} \right) \quad (1)$$

Note that the derived unit of charge $\sqrt{\hbar/Z_0}$ is independent of T

Ⓒ $[M^{-1}L^3T^{-2}]$ is the rationalized universal constant of gravitation – the row vector of the exponents of $1 \text{ M}^{-1} \cdot \text{L}^3 \cdot \text{T}^{-2}$ right multiplied by the inverted matrix of (1) yields a derivation of the unit

$$\left[\begin{array}{cc|ccc} \text{Q} & \Theta & \text{M} & \text{L} & \text{T} \\ 0 & 0 & -1 & 3 & -2 \end{array} \right] \stackrel{(1)}{\mapsto} \left[\begin{array}{cc|ccc} Z_0 & k & \hbar & c & T \\ 0 & 0 & -1 & 5 & 2 \end{array} \right] \Rightarrow 1 \text{ M}^{-1} \cdot \text{L}^3 \cdot \text{T}^{-2} = \frac{c^5 T^2}{\hbar} \quad \therefore \frac{\text{Ⓒ}}{1 \text{ M}^{-1} \cdot \text{L}^3 \cdot \text{T}^{-2}} = \left(\frac{T}{\sqrt{\hbar \text{Ⓒ}/c^5}} \right)^{-2} \quad (2)$$

$T = \sqrt{\hbar \text{Ⓒ}/c^5}$ for Wholly Derived System of Units

$$\left. \begin{aligned} Z_0 &= 1 \text{ Q}^{-2} \cdot \text{M} \cdot \text{L}^2 \cdot \text{T}^{-1} & 1 \text{ Q} &= \sqrt{\hbar/Z_0} \\ k &= 1 \text{ } \Theta^{-1} \cdot \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} & 1 \text{ } \Theta &= \frac{1}{k} \sqrt{\hbar c^5/\text{Ⓒ}} \\ \hbar &= 1 \text{ M} \cdot \text{L}^2 \cdot \text{T}^{-1} & \Leftrightarrow 1 \text{ M} &= \sqrt{\hbar c/\text{Ⓒ}} \\ c &= 1 \text{ L} \cdot \text{T}^{-1} & 1 \text{ L} &= \sqrt{\hbar \text{Ⓒ}/c^3} \\ \text{Ⓒ} &= 1 \text{ M}^{-1} \cdot \text{L}^3 \cdot \text{T}^{-2} & 1 \text{ T} &= \sqrt{\hbar \text{Ⓒ}/c^5} \end{aligned} \right\} \left(\begin{array}{ccc|ccc} -2 & 0 & 1 & 2 & -1 & \\ 0 & -1 & 1 & 2 & -2 & \\ & & 1 & 2 & -1 & \\ 0 & & 0 & 1 & -1 & \\ & & & -1 & 3 & -2 \end{array} \right)^{-1} = \left(\begin{array}{ccc|ccc} -1/2 & 0 & 1/2 & 0 & 0 & \\ 0 & -1 & 1/2 & 5/2 & -1/2 & \\ & & 1/2 & 1/2 & -1/2 & \\ & 0 & 1/2 & -3/2 & 1/2 & \\ & & 1/2 & -5/2 & 1/2 & \end{array} \right) \quad (3)$$

Relative Error in Conversions from SI and CODATA 2022

SI	relative error	via (1)	relative error	via (3)	relative error
Z_0	1.50×10^{-10}	1 Q	7.50×10^{-11}	1 Q	7.50×10^{-11}
k	exactly	1 Θ	exactly*	1 Θ	1.10×10^{-5}
\hbar	exactly	1 M	exactly*	1 M	1.10×10^{-5}
c	exactly	1 L	exactly*	1 L	1.10×10^{-5}
Ⓒ	2.20×10^{-5}	1 T	exactly*	1 T	1.10×10^{-5}

*whenever $\frac{T}{1\text{s}}$ is exact

$$\begin{aligned} k &= 13.80649 \times 10^{-24} \text{ J/K} \\ \hbar &= 662.607015 \times 10^{-36} \text{ J}\cdot\text{s} \\ c &= 299\,792\,458 \text{ m/s} \\ Z_0 &= 376.730\,313\,412 \text{ J}\cdot\text{s}/c^2 \\ &\quad \pm 0.000\,000\,059 \text{ J}\cdot\text{s}/c^2 (\Omega) \\ G &= 66.7430 \times 10^{-12} \text{ N}\cdot\text{m}^2/\text{kg}^2 \\ &\quad \pm 0.0015 \times 10^{-12} \text{ N}\cdot\text{m}^2/\text{kg}^2 \end{aligned}$$

Practically-Sized Units from Improbably-Sized Planck Scale Units

★ $T = \sqrt{\hbar G/c^5}$ for ^(1&3) \mapsto **Planck Scale Wholly Derived System of Units (p – thorn)**

$$\begin{aligned}
 3.767 \times 10^2 \text{ J}\cdot\text{s}/c^2 &= Z_0 = 1 \text{ pe}\cdot\text{pt}/\text{pq}^2 & 1 \text{ pq} &= 5.291 \times 10^{-19} \text{ C} \\
 1.381 \times 10^{-23} \text{ J}/\text{K} &= k = 1 \text{ pe}/\text{pa} & (3) \quad 1 \text{ pa} &= 3.997 \times 10^{31} \text{ K} \\
 1.055 \times 10^{-34} \text{ J}\cdot\text{s} &= \hbar = 1 \text{ pe}\cdot\text{pt} & \Rightarrow \quad 1 \text{ pm} &= 6.140 \times 10^{-9} \text{ kg} \\
 2.998 \times 10^8 \text{ m}/\text{s} &= c = 1 \text{ pl}/\text{pt} & 1 \text{ pl} &= 5.729 \times 10^{-35} \text{ m} \\
 8.387 \times 10^{-10} \text{ N}\cdot\text{m}^2/\text{kg}^2 &= G = 1 \text{ pf}\cdot\text{pl}^2/\text{pm}^2 & 1 \text{ pt} &= 1.911 \times 10^{-43} \text{ s}
 \end{aligned}$$

e.g.: a Required Multiple of the Planck Time for a Unit on the Order of a Second or a Day	\Rightarrow	Practically-Sized Units for																																							
<table border="1"> <tr> <td></td> <td>\log_{60}</td> <td>\log_{10}</td> <td>\log_2</td> <td></td> </tr> <tr> <td>$1 \text{ s}/1 \text{ pt}$</td> <td>24.02</td> <td>42.72</td> <td>141.91</td> <td></td> </tr> <tr> <td>$1 \text{ day}/1 \text{ pt}$</td> <td>26.80</td> <td>47.66</td> <td>158.31</td> <td></td> </tr> </table>				\log_{60}	\log_{10}	\log_2		$1 \text{ s}/1 \text{ pt}$	24.02	42.72	141.91		$1 \text{ day}/1 \text{ pt}$	26.80	47.66	158.31		<table border="1"> <tr> <td>charge [q]</td> <td>F● = 60^9</td> <td>10^{16}</td> <td>2^{53}</td> <td>p</td> </tr> <tr> <td>temperature [a]</td> <td>°● = 60^{-18}</td> <td>10^{-32}</td> <td>2^{-106}</td> <td>p</td> </tr> <tr> <td>mass [m]</td> <td>q● = 60^4</td> <td>10^7</td> <td>2^{24}</td> <td>p</td> </tr> <tr> <td>length [l]</td> <td>f● = 60^{19}</td> <td>10^{34}</td> <td>2^{112}</td> <td>p</td> </tr> <tr> <td>time [t]</td> <td>h● = 60^{24}</td> <td>10^{43}</td> <td>2^{142}</td> <td>p</td> </tr> </table>	charge [q]	F● = 60^9	10^{16}	2^{53}	p	temperature [a]	°● = 60^{-18}	10^{-32}	2^{-106}	p	mass [m]	q● = 60^4	10^7	2^{24}	p	length [l]	f● = 60^{19}	10^{34}	2^{112}	p	time [t]	h● = 60^{24}	10^{43}
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★★ T as fractions of $1 \text{ s} \approx \sqrt{\hbar G/c^5}$ for ⁽¹⁾ \mapsto **Planck Scale Time Derived Units (pf, pd and pb)**

1 pft = 60^{-24} s for Units (pf) **1 pdt = 10^{-43} s for Units (pd)** **1 pbt = 2^{-142} s for Units (pb)**

$1 \text{ pfa} = 3.619 \times 10^{31} \text{ K}$	$1 \text{ pda} = 7.638 \times 10^{31} \text{ K}$	$1 \text{ pba} = 4.258 \times 10^{31} \text{ K}$
$1 \text{ pfm} = 5.560 \times 10^{-9} \text{ kg}$	$1 \text{ pdm} = 1.173 \times 10^{-8} \text{ kg}$	$1 \text{ pbm} = 6.542 \times 10^{-9} \text{ kg}$
$1 \text{ pfl} = 6.327 \times 10^{-35} \text{ m}$	$1 \text{ pdl} = 2.998 \times 10^{-35} \text{ m}$	$1 \text{ pbl} = 5.377 \times 10^{-35} \text{ m}$
$1 \text{ pft} = 2.110 \times 10^{-43} \text{ s}$	$1 \text{ pdt} = 1.000 \times 10^{-43} \text{ s}$	$1 \text{ pbt} = 1.794 \times 10^{-43} \text{ s}$
$\frac{G}{1 \text{ pff}\cdot\text{pft}^2/\text{pfm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pft}}{1 \text{ pt}}\right)^{-2} = 0.820 \text{ 1}$	$\frac{G}{1 \text{ pdf}\cdot\text{pdt}^2/\text{pdm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pdt}}{1 \text{ pt}}\right)^{-2} = 3.652$	$\frac{G}{1 \text{ pbf}\cdot\text{pbt}^2/\text{pbm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pbt}}{1 \text{ pt}}\right)^{-2} = 1.135$

★★★ T as fractions of $1 \text{ day} \approx \sqrt{\hbar G/c^5}$ for **Planck Scale Time Derived Units (pß, pdd and pbb)**

1 pßt = 60^{-27} day for (pß) **1 pddt = 10^{-48} day for (pdd)** **1 pbbt = 2^{-158} day for (pbb)**

$1 \text{ pßa} = 9.048 \times 10^{31} \text{ K}$	$1 \text{ pdda} = 8.841 \times 10^{31} \text{ K}$	$1 \text{ pbbba} = 3.230 \times 10^{31} \text{ K}$
$1 \text{ pßm} = 1.390 \times 10^{-8} \text{ kg}$	$1 \text{ pddm} = 1.358 \times 10^{-8} \text{ kg}$	$1 \text{ pbbm} = 4.962 \times 10^{-9} \text{ kg}$
$1 \text{ pßl} = 2.531 \times 10^{-35} \text{ m}$	$1 \text{ pddl} = 2.590 \times 10^{-35} \text{ m}$	$1 \text{ pbbbl} = 7.089 \times 10^{-35} \text{ m}$
$1 \text{ pßt} = 8.442 \times 10^{-44} \text{ s}$	$1 \text{ pddt} = 8.640 \times 10^{-44} \text{ s}$	$1 \text{ pbbt} = 2.365 \times 10^{-43} \text{ s}$
$\frac{G}{1 \text{ pßf}\cdot\text{pßt}^2/\text{pßm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pßt}}{1 \text{ pt}}\right)^{-2} = 5.125$	$\frac{G}{1 \text{ pddf}\cdot\text{pddt}^2/\text{pddm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pddt}}{1 \text{ pt}}\right)^{-2} = 4.893$	$\frac{G}{1 \text{ pbbf}\cdot\text{pbbt}^2/\text{pbbm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ pbbt}}{1 \text{ pt}}\right)^{-2} = 0.653 \text{ 2}$

Shifted (p, pf and pß) Units

$$\begin{aligned}
 1 \text{ Ffq} &= 60^9 \text{ pq} = 5.332 \text{ mC} \\
 1 \text{ °fa} &= 60^{-18} \text{ pa} = 0.393 \text{ 5 K} \\
 1 \text{ qfm} &= 60^4 \text{ pm} = 79.57 \text{ g} \\
 1 \text{ ffl} &= 60^{19} \text{ pl} = 0.349 \text{ 1 m} \\
 1 \text{ hft} &= 60^{24} \text{ pt} = 0.905 \text{ 6 s} \\
 1 \text{ °ßa} &= 60^{-18} \text{ pfa} = 0.356 \text{ 4 K} \\
 1 \text{ qßm} &= 60^4 \text{ pfm} = 72.06 \text{ g} \\
 1 \text{ fßl} &= 60^{19} \text{ pfl} = 0.385 \text{ 5 m} \\
 1 \text{ hßt} &= 60^{24} \text{ pft} = 1 \text{ s} \\
 1 \text{ °fßa} &= 60^{-18} \text{ pßa} = 0.890 \text{ 9 K} \\
 1 \text{ qfßm} &= 60^4 \text{ pßm} = 180.1 \text{ g} \\
 1 \text{ ffßl} &= 60^{19} \text{ pßl} = 0.154 \text{ 2 m} \\
 1 \text{ hfßt} &= 60^{24} \text{ pßt} = 0.400 \text{ 0 s}
 \end{aligned}$$

Shifted (p, pd and pdd) Units

$$\begin{aligned}
 1 \text{ Fdq} &= 10^{16} \text{ pq} = 5.291 \text{ mC} \\
 1 \text{ °da} &= 10^{-32} \text{ pa} = 0.399 \text{ 7 K} \\
 1 \text{ qdm} &= 10^7 \text{ pm} = 61.40 \text{ g} \\
 1 \text{ fdl} &= 10^{34} \text{ pl} = 0.572 \text{ 9 m} \\
 1 \text{ hdt} &= 10^{43} \text{ pt} = 1.911 \text{ s} \\
 1 \text{ °dda} &= 10^{-32} \text{ pda} = 0.763 \text{ 8 K} \\
 1 \text{ qddm} &= 10^7 \text{ pdm} = 117.3 \text{ g} \\
 1 \text{ fddl} &= 10^{34} \text{ pdl} = 0.299 \text{ 8 m} \\
 1 \text{ hddt} &= 10^{43} \text{ pdt} = 1 \text{ s} \\
 1 \text{ °ddda} &= 10^{-32} \text{ pdda} = 0.884 \text{ 1 K} \\
 1 \text{ qdddm} &= 10^7 \text{ pddm} = 135.8 \text{ g} \\
 1 \text{ fdddl} &= 10^{34} \text{ pddl} = 0.259 \text{ 0 m} \\
 1 \text{ hdddt} &= 10^{43} \text{ pddt} = 0.864 \text{ 0 s}
 \end{aligned}$$

Shifted (p, pb and pbb) Units

$$\begin{aligned}
 1 \text{ Fbq} &= 2^{53} \text{ pq} = 4.766 \text{ mC} \\
 1 \text{ °ba} &= 2^{-106} \text{ pa} = 0.492 \text{ 6 K} \\
 1 \text{ qbm} &= 2^{24} \text{ pm} = 103.0 \text{ g} \\
 1 \text{ fbl} &= 2^{112} \text{ pl} = 0.297 \text{ 5 m} \\
 1 \text{ hbt} &= 2^{142} \text{ pt} = 1.065 \text{ 5 s} \\
 1 \text{ °bba} &= 2^{-106} \text{ pba} = 0.524 \text{ 9 K} \\
 1 \text{ qbbm} &= 2^{24} \text{ pbm} = 109.8 \text{ g} \\
 1 \text{ fbbbl} &= 2^{112} \text{ pbl} = 0.279 \text{ 2 m} \\
 1 \text{ hbbt} &= 2^{142} \text{ pbt} = 1 \text{ s} \\
 1 \text{ °bbba} &= 2^{-106} \text{ pbba} = 0.398 \text{ 1 K} \\
 1 \text{ qbbbm} &= 2^{24} \text{ pbbm} = 83.25 \text{ g} \\
 1 \text{ fbbbl} &= 2^{112} \text{ pbbbl} = 0.368 \text{ 1 m} \\
 1 \text{ hbbbt} &= 2^{142} \text{ pbbt} = 1.318 \text{ s}
 \end{aligned}$$

Fundamental Constants in Systems Shifted from the Planck Scale

The fundamental constants of nature Z_0, k, \hbar, c and, for the wholly derived units, \mathbb{G} will be represented by whole powers of the respective number base for these sexagenary (f), denary (d) and binary (b) shifted units. When T only approximates $\sqrt{\hbar\mathbb{G}/c^5}$ then, in the scientific notation for its own number base, \mathbb{G} will have a small significand as determined by equation (2).

$$\left. \begin{array}{l} \mathbf{F}\bullet = 60^9 \quad 10^{16} \quad 2^{53} \quad \mathbf{p} \quad [\mathbf{Q}] \\ \circ\bullet = 60^{-18} \quad 10^{-32} \quad 2^{-106} \quad \mathbf{p} \quad [\Theta] \\ \mathbf{q}\bullet = 60^4 \quad 10^7 \quad 2^{24} \quad \mathbf{p} \quad [\mathbf{M}] \\ \mathbf{f}\bullet = 60^{19} \quad 10^{34} \quad 2^{112} \quad \mathbf{p} \quad [\mathbf{L}] \\ \mathbf{h}\bullet = 60^{24} \quad 10^{43} \quad 2^{142} \quad \mathbf{p} \quad [\mathbf{T}] \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Z_0 [\mathbf{Q}^{-2}\mathbf{M}\mathbf{L}^2\mathbf{T}^{-1}] = 1 \quad 1 \quad 1 \quad \text{erg}\bullet\text{e}\cdot\mathbf{h}\bullet\mathbf{t}/\mathbf{F}\bullet\mathbf{q}^2 \\ k [\Theta^{-1}\mathbf{M}\mathbf{L}^2\mathbf{T}^{-2}] = 60^{-12} \quad 10^{-21} \quad 2^{-70} \quad \text{erg}\bullet\text{e}/\circ\bullet\mathbf{a} \\ \hbar [\mathbf{M}\mathbf{L}^2\mathbf{T}^{-1}] = 60^{-18} \quad 10^{-32} \quad 2^{-106} \quad \text{erg}\bullet\text{e}\cdot\mathbf{h}\bullet\mathbf{t} \\ c [\mathbf{L}\mathbf{T}^{-1}] = 60^5 \quad 10^9 \quad 2^{30} \quad \mathbf{f}\bullet/\mathbf{h}\bullet\mathbf{t} \\ \mathbb{G} [\mathbf{M}^{-1}\mathbf{L}^3\mathbf{T}^{-2}] \approx 60^{-5} \quad 10^{-9} \quad 2^{-28} \quad \text{dyn}\bullet\mathbf{f}\cdot\mathbf{f}\bullet\mathbf{l}^2/\mathbf{q}\bullet\mathbf{m}^2 \end{array} \right.$$

So while $\mathbb{G} = 60^{-5} \text{ dyn}\bullet\mathbf{f}\cdot\mathbf{f}\bullet\mathbf{l}^2/\mathbf{q}\bullet\mathbf{m}^2$ exactly in the wholly derived sexagenary units in the similar time derived units we have $\mathbb{G} = 0.8201 \times 60^{-5} \text{ dyn}\bullet\mathbf{f}\cdot\mathbf{f}\bullet\mathbf{l}^2/\mathbf{q}\bullet\mathbf{m}^2$ and $\mathbb{G} = 5.125 \times 60^{-5} \text{ dyn}\bullet\mathbf{f}\cdot\mathbf{f}\bullet\mathbf{l}^2/\mathbf{q}\bullet\mathbf{m}^2$.

Practically-Sized Constants Leading to Practically-Sized Units

The constant of nature Z_0 would already derive a practically sized unit. Dividing other constants $k, \hbar = h/2\pi, c$ and, for a wholly derived system, $\mathbb{G} = 4\pi G$ by powers of 60, 10 or 2 before calculating the wholly derived (via (3)) or the $T=1$ s derived (via (1)) base units directly yields sexagenary, denary and binary units at a convenient size with no further scaling needed; Dividing $T=1$ day into parts via a power of the base, rather than the using the second directly, better honours the numbering system; the divided day yields time derived (via (1)) base units in systems where hours, minutes and seconds make little sense. For example, the 24 hour day can be divided in $10 \times 100 \times 100$ parts yielding a conveniently-sized unit of time that is a little bit shorter than a second.

For example, dividing c by 10^9 before the derivation yields units T and L such that $c = 1 \times 10^9 \text{ L}\cdot\mathbf{T}^{-1}$.

Weirder powers of 20, 12 or 3 would yield vicenary (ϕ), duodenary (\mathbf{d}) and ternary (θ) systems.

foundational constants								divisors		
Z_0	\div	1	1	1	1	1	1	} e.g. interpreting the table for wholly derived denary units would result in the system where		
k	\div	60^{-12}	10^{-21}	2^{-70}	20^{-16}	12^{-20}	3^{-44}			
\hbar	\div	60^{-18}	10^{-32}	2^{-106}	20^{-24}	12^{-30}	3^{-67}			
c	\div	60^5	10^9	2^{30}	20^7	12^8	3^{19}			
\mathbb{G}	\div	60^{-5}	10^{-9}	2^{-28}	20^{-7}	12^{-9}	3^{-17}			
1 s	\div	1	1	1	1	1	1	} $Z_0 = 1 \text{ erg}\bullet\text{e}\cdot\mathbf{h}\bullet\mathbf{t}/\mathbf{F}\bullet\mathbf{d}\mathbf{q}^2$ $k = 1 \times 10^{-21} \text{ erg}\bullet\text{e}/\circ\bullet\mathbf{a}$ $\hbar = 1 \times 10^{-32} \text{ erg}\bullet\text{e}\cdot\mathbf{h}\bullet\mathbf{t}$ $c = 1 \times 10^9 \text{ f}\bullet\mathbf{l}/\mathbf{h}\bullet\mathbf{t}$ $\mathbb{G} = 1 \times 10^{-9} \text{ dyn}\bullet\mathbf{f}\cdot\mathbf{f}\bullet\mathbf{l}^2/\mathbf{q}\bullet\mathbf{d}\mathbf{m}^2$		
1 day	\div	60^3	10^5	2^{16}	20^4	12^5	3^{10}			

Parity and the Ratio between the Practically-Sized and Planck Scale Units

A parity relation on the number base logarithm of the divisor for \hbar determines whether the conveniently sized charge unit derived this way is rationally commensurable with the Planck scale charge unit; an even logarithm leads to rationally commensurable charge units, an odd logarithm does not and introduces a factor of the square root of the number base.

A parity relation on the sum of logarithms of the divisors for \hbar, c and \mathbb{G} determines whether the wholly determined and conveniently sized units of time, distance, mass and temperature are rationally commensurable with their respective Planck scale units; an even sum of logarithms leads to rationally commensurable units.

Having our practically-sized system of units be rationally commensurable with Planck scale units seems desirable (and given its derivation is straightforward) but would only rarely be a useful.

The divisors for the number base 60, 10 and base 2 case come from their scaled cousins described earlier in this document and so automatically satisfy the parity relation; in this case, the scaled Planck units are necessarily the same as the units directly derived from the divided constants. The vicenary (number base 20) case is also well behaved and could have defined by a shift from the Planck scale.

The number base 12 and base 3 units work better when the divided constants result in an odd parity relation and the practically derived units are thus rationally incommensurable with the Planck units. The freedom to shift the derived duodenary units by $\sqrt{12}$ and the ternary units by $\sqrt{3}$ is more useful.

e.g. Denary Systems of Units (d, dd and ddd)

denary divisors	
$Z_0 \div 1$	$= 376.7 \text{ J}\cdot\text{s}/\text{C}^2$
$k \div 10^{-21}$	$= 0.01381 \text{ J}/\text{K}$
$\hbar \div 10^{-32}$	$= 0.01055 \text{ J}\cdot\text{s}$
$c \div 10^9$	$= 0.2998 \text{ m}/\text{s}$
$\mathbb{G} \div 10^{-9}$	$= 0.8387 \text{ N}\cdot\text{m}^2/\text{kg}^2$
$1 \text{ s} \div 1$	$= 1 \text{ s}$
$1 \text{ day} \div 10^5$	$= 0.8640 \text{ s}$

time	length	mass	force	work	temp.	charge
hdt	fdl	qdm	dyndf	ergde	°da	Fdq
hddt	fddl	qddm	dynddf	ergdde	°dda	Fddq
hddd	fdddl	qdddm	dyndddf	ergddde	°ddda	Fdddq

T parameter for the (1) transformation

$$1 \text{ hdt} = \sqrt{\frac{(0.01055 \text{ J}\cdot\text{s}) \cdot (0.8387 \text{ N}\cdot\text{m}^2/\text{kg}^2)}{(0.2998 \text{ m}/\text{s})^5}} = 1.911 \text{ s}$$

$$1 \text{ hddt} = 1 \text{ s}$$

$$1 \text{ hddd} = 10^{-5} \text{ day} = \frac{24 \times 60 \times 60 \text{ s}}{10 \times 100 \times 100} = \frac{86400 \text{ s}}{100000} = 0.8640 \text{ s}$$

Note that feeding the parameter 1 hdt into the (1) transformation is the same as running the (3) transformation. Note also that $\frac{1 \text{ hddd}}{1 \text{ hddt}} = \frac{10^{-5} \text{ day}}{1 \text{ s}} = \frac{1 \text{ hddt}}{1 \text{ hdt}}$ ensuring consistency with the previous derivation of the ddd units as being shifted from the Planck scale. We express inputs and outputs to the transformations in fully coherent units (without SI prefixed forms) for clarity in numerical calculation.

$$\begin{array}{llll} Z_0 \div 1 & = 376.7 \text{ J}\cdot\text{s}/\text{C}^2 & = 1 \text{ ergde}\cdot\text{hdt}/\text{Fdq}^2 & 1 \text{ Fdq} = 0.005291 \text{ C} \\ k \div 10^{-21} & = 0.01381 \text{ J}/\text{K} & = 1 \text{ ergde}/\text{°da} & 1 \text{ °da} = 0.3997 \text{ K} \\ \hbar \div 10^{-32} & = 0.01055 \text{ J}\cdot\text{s} & = 1 \text{ ergde}\cdot\text{hdt} & \Rightarrow 1 \text{ qdm} = 0.06140 \text{ kg} \\ c \div 10^9 & = 0.2998 \text{ m}/\text{s} & = 1 \text{ fdl}/\text{hdt} & 1 \text{ fdl} = 0.5729 \text{ m} \\ \mathbb{G} \div 10^{-9} & = 0.8387 \text{ N}\cdot\text{m}^2/\text{kg}^2 & = 1 \text{ dyndf}\cdot\text{fdl}^2/\text{qdm}^2 & 1 \text{ hdt} = 1.911 \text{ s} \end{array}$$

$$\begin{array}{llll} Z_0 \div 1 & = 376.7 \text{ J}\cdot\text{s}/\text{C}^2 & = 1 \text{ ergdde}\cdot\text{hddt}/\text{Fddq}^2 & 1 \text{ Fddq} = 0.005291 \text{ C} \\ k \div 10^{-21} & = 0.01381 \text{ J}/\text{K} & = 1 \text{ ergdde}/\text{°dda} & 1 \text{ °dda} = 0.7638 \text{ K} \\ \hbar \div 10^{-32} & = 0.01055 \text{ J}\cdot\text{s} & = 1 \text{ ergdde}\cdot\text{hddt} & \Rightarrow 1 \text{ qddm} = 0.1173 \text{ kg} \\ c \div 10^9 & = 0.2998 \text{ m}/\text{s} & = 1 \text{ fddl}/\text{hddt} & 1 \text{ fddl} = 0.2998 \text{ m} \\ 1 \text{ s} \div 1 & = 1 \text{ s} & = 1 \text{ hddt} & 1 \text{ hddt} = 1 \text{ s} \\ & & \therefore \frac{\mathbb{G} \div 10^{-9}}{1 \text{ dynddf}\cdot\text{fddl}^2/\text{qddm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ hddt}}{1 \text{ hdt}}\right)^{-2} = \left(\frac{1 \text{ hdt}}{1 \text{ hddt}}\right)^2 = 3.652 \end{array}$$

$$\begin{array}{llll} Z_0 \div 1 & = 376.7 \text{ J}\cdot\text{s}/\text{C}^2 & = 1 \text{ ergddd}\cdot\text{hddd}/\text{Fdddq}^2 & 1 \text{ Fdddq} = 0.005291 \text{ C} \\ k \div 10^{-21} & = 0.01381 \text{ J}/\text{K} & = 1 \text{ ergddd}/\text{°ddda} & 1 \text{ °ddda} = 0.8841 \text{ K} \\ \hbar \div 10^{-32} & = 0.01055 \text{ J}\cdot\text{s} & = 1 \text{ ergddd}\cdot\text{hddd} & \Rightarrow 1 \text{ qdddm} = 0.1358 \text{ kg} \\ c \div 10^9 & = 0.2998 \text{ m}/\text{s} & = 1 \text{ fddd}/\text{hddd} & 1 \text{ fddd} = 0.2590 \text{ m} \\ 1 \text{ day} \div 10^5 & = 0.8640 \text{ s} & = 1 \text{ hddd} & 1 \text{ hddd} = 0.8640 \text{ s} \\ & & \therefore \frac{\mathbb{G} \div 10^{-9}}{1 \text{ dyndddf}\cdot\text{fddd}^2/\text{qdddm}^2} \stackrel{(2)}{=} \left(\frac{1 \text{ hddd}}{1 \text{ hdt}}\right)^{-2} = \left(\frac{1 \text{ hdt}}{1 \text{ hddd}}\right)^2 = 4.893 \end{array}$$

We have repeated the charge units $1 \text{ Fdq} = 1 \text{ Fddq} = 1 \text{ Fdddq}$ which are independent of the time parameter in the (1) transformation; otherwise these are exactly the same values as we calculated by shifting from the Planck scale. We would do well to follow our former presentation by using prefixed SI forms for the output units (certainly the mC and possibly the g) for a more readable presentation of the numbers.

Comparing Three Ways of Deriving Practically Sized Units

Together with the wholly derived Planck scale system of units \mathfrak{p} we will also define two systems of time derived units \mathfrak{d} and \mathfrak{dd} at no sensible scale. We will use these as intermediaries between our derivation from fundamental constants divided by powers of the number base and the shifted units we first investigated.

$T = 1 \text{ s}$ for Time Derived System of Units (\mathfrak{d} – eth) at No Sensible Scale

$$\begin{aligned} Z_0 &= 1 \frac{\delta e \cdot \delta t}{\delta q^2} & 1 \delta q &= 5.291 \times 10^{-19} \text{ C} \\ k &= 1 \frac{\delta e}{\delta a} & 1 \delta a &= 7.638 \times 10^{-12} \text{ K} \\ \hbar &= 1 \delta e \cdot \delta t & \stackrel{(1)}{\Rightarrow} 1 \delta m &= 1.173 \times 10^{-51} \text{ kg} \\ c &= 1 \frac{\delta l}{\delta t} & 1 \delta l &= 2.998 \times 10^8 \text{ m} \\ 1 \text{ s} &= 1 \delta t & 1 \delta t &= 1 \text{ s} \end{aligned}$$

$T = 1 \text{ day}$ for Time Derived System of Units (\mathfrak{dd}) at No Sensible Scale

$$\begin{aligned} Z_0 &= 1 \frac{\delta \delta e \cdot \delta \delta t}{\delta \delta q^2} & 1 \delta \delta q &= 5.291 \times 10^{-19} \text{ C} \\ k &= 1 \frac{\delta \delta e}{\delta \delta a} & 1 \delta \delta a &= 8.841 \times 10^{-17} \text{ K} \\ \hbar &= 1 \delta \delta e \cdot \delta \delta t & \stackrel{(1)}{\Rightarrow} 1 \delta \delta m &= 1.358 \times 10^{-56} \text{ kg} \\ c &= 1 \frac{\delta \delta l}{\delta \delta t} & 1 \delta \delta l &= 2.590 \times 10^{13} \text{ m} \\ 1 \text{ day} &= 1 \delta \delta t & 1 \delta \delta t &= 8.640 \times 10^4 \text{ s} \end{aligned}$$

Note that the \mathfrak{d} and \mathfrak{dd} system of units are rationally commensurable via a factor of $86400 = 24 \times 60 \times 60$. The length $1 \delta l$ is a lightsecond and $1 \delta \delta l$ a lightday; only the time and length units merit having names. These two systems of units are of no great interest by themselves but do allow us the following reckoning. Starting with the desired form for the fundamental constants we can use: the (3) transformation to back derive to the already known values of \mathfrak{p} system; and the (1) transformation to back derive to the already known values of the \mathfrak{d} and \mathfrak{dd} systems. This works in any number base but, taking as a specific example, the denary system \mathfrak{d} is shifted from the \mathfrak{p} system, the denary system \mathfrak{dd} is shifted from the \mathfrak{d} system and the denary system \mathfrak{ddd} is shifted from the \mathfrak{dd} system.

$$\begin{aligned} 1 \text{ ergde} \cdot \text{hdt} / \text{Fdq}^2 &= Z_0 = 1 \mathfrak{p} e \cdot \mathfrak{p} t / \mathfrak{p} q^2 & 1 \mathfrak{p} q &= 10^{-16} \text{ Fdq} \\ 10^{-21} \text{ ergde} / \text{o} \text{da} &= k = 1 \mathfrak{p} e / \mathfrak{p} a & 1 \mathfrak{p} a &= 10^{32} \text{ o} \text{da} \\ 10^{-32} \text{ ergde} \cdot \text{hdt} &= \hbar = 1 \mathfrak{p} e \cdot \mathfrak{p} t & \stackrel{(3)}{\Rightarrow} 1 \mathfrak{p} m &= 10^{-7} \text{ qdm} \\ 10^9 \text{ fdl} / \text{hdt} &= c = 1 \mathfrak{p} l / \mathfrak{p} t & 1 \mathfrak{p} l &= 10^{-34} \text{ fdl} \\ 10^{-9} \text{ dyndf} \cdot \text{fdl}^2 / \text{qdm}^2 &= \mathbb{G} = 1 \mathfrak{p} f \cdot \mathfrak{p} l^2 / \mathfrak{p} m^2 & 1 \mathfrak{p} t &= 10^{-43} \text{ hdt} \end{aligned} \quad \begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & -1 & 1/2 & 5/2 & -1/2 \\ & & 1/2 & 1/2 & -1/2 \\ 0 & & 1/2 & -3/2 & 1/2 \\ & & & 1/2 & -5/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ -9 \end{bmatrix} = \begin{bmatrix} -16 \\ 32 \\ -7 \\ -34 \\ -43 \end{bmatrix}$$

$$\begin{aligned} 1 \text{ ergdde} \cdot \text{s} / \text{Fdddq}^2 &= Z_0 = 1 \delta e \cdot \delta t / \delta q^2 & 1 \delta q &= 10^{-16} \text{ Fddq} \\ 10^{-21} \text{ ergdde} / \text{o} \text{dda} &= k = 1 \delta e / \delta a & 1 \delta a &= 10^{-11} \text{ o} \text{dda} \\ 10^{-32} \text{ ergdde} \cdot \text{hddt} &= \hbar = 1 \delta e \cdot \delta t & \stackrel{(1)}{\Rightarrow} 1 \delta m &= 10^{-50} \text{ qddm} \\ 10^9 \text{ fddl} / \text{hddt} &= c = 1 \delta l / \delta t & 1 \delta l &= 10^9 \text{ fddl} \\ 1 \text{ hddt} &= T = 1 \delta t & 1 \delta t &= 1 \text{ hddt} \end{aligned} \quad \begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ & & 1 & -2 & -1 \\ 0 & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} -16 \\ -11 \\ -50 \\ 9 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 1 \text{ ergddd} \cdot \text{s} / \text{Fdddq}^2 &= Z_0 = 1 \delta \delta e \cdot \text{s} / \delta \delta q^2 & 1 \delta \delta q &= 10^{-16} \text{ Fdddq} \\ 10^{-21} \text{ ergddd} / \text{o} \text{ddda} &= k = 1 \delta \delta e / \delta \delta a & 1 \delta \delta a &= 10^{-16} \text{ o} \text{ddda} \\ 10^{-32} \text{ ergddd} \cdot \text{hddd} &= \hbar = 1 \delta \delta e \cdot \delta \delta t & \stackrel{(1)}{\Rightarrow} 1 \delta \delta m &= 10^{-55} \text{ qdddm} \\ 10^9 \text{ fddd} / \text{hddd} &= c = 1 \delta \delta l / \delta \delta t & 1 \delta \delta l &= 10^{14} \text{ fddd} \\ 10^5 \text{ hddd} &= T = 1 \delta \delta t & 1 \delta \delta t &= 10^5 \text{ hddd} \end{aligned} \quad \begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \\ & -1 & 1 & 0 & -1 \\ & & 1 & -2 & -1 \\ 0 & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \\ -32 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} -16 \\ -16 \\ -55 \\ 14 \\ 5 \end{bmatrix}$$

So we simply divide the known values of the \mathfrak{p} , \mathfrak{d} and \mathfrak{dd} by their respective derived powers to realize the units of the \mathfrak{d} , \mathfrak{dd} and \mathfrak{ddd} systems. This is more complicated than the direct derivation of the denary units from the numeric value of fundamental constants in the desired form but it does serve to show the hidden connection between systems tuned to different number bases.

This presentation also makes clear how half integral exponents can appear when the parity relation is odd; such shifts work perfectly well but lead to rationally incommensurable units by definition.

Our initial method of deriving the time derived denary units from base ten friendly approximations $1 \mathfrak{p} \text{dt}$ and $1 \mathfrak{p} \text{d} \text{dt}$ of the Planck scale $1 \mathfrak{p} \text{t}$ simultaneously scales the systems $\mathfrak{p} \rightarrow \mathfrak{d}$, $\mathfrak{p} \text{d} \rightarrow \mathfrak{dd}$ and $\mathfrak{p} \text{dd} \rightarrow \mathfrak{ddd}$. This needs to be consistent with the base ten friendly division of a day $1 \text{ hddd} \text{t}$ as an approximation of $1 \text{ hddt} = 1 \text{ s}$. There is no associativity that assures the best approximations require $\frac{1 \mathfrak{p} \text{d} \text{dt}}{1 \mathfrak{p} \text{dt}} = \frac{1 \text{ hddd} \text{t}}{1 \text{ hddt}}$ but as long as it does hold all our three different derivations for time derived units must agree with each other.

Chosen Coherence in Reference Units for a Nice Presentation

We have chosen to size our derived natural units with a unit of mass somewhat less than a kilogramme and a unit of charge much less than a coulomb. This makes it tricky to express the values of the conversions between systems in a numeric table which directly express the (1) and (3) transformations while avoiding scientific notation for those numbers. We can state the fundamental constants of nature not in SI, but in the coherent system of second-metre-gramme-kelvin-millicoulomb units by juggling a power of a thousand here or there; After applying our transformations we get a nicely scaled table of numbers which needs little or no formatting

Conversion from Coherent Second-Metre-Gramme-Kelvin-Millicoulomb Units

Z_0	0.376 730 313 412			$\text{mC}^{-2} \cdot \text{g} \cdot \text{m}^2 \cdot \text{s}^{-1}$
	sexagenary β, f	denary d	binary b	
k	$30.053\,723\,554 \times 60^{-12}$	$13.806\,490\,000 \times 10^{-21}$	$16.299\,826\,406 \times 2^{-70}$	$\text{K}^{-1} \cdot \text{g} \cdot \text{m}^2 \cdot \text{s}^{-2}$
h	$10.710\,226\,810 \times 60^{-18}$	$10.545\,718\,176 \times 10^{-32}$	$8.555\,703\,025 \times 2^{-106}$	$\text{g} \cdot \text{m}^2 \cdot \text{s}^{-1}$
c	$0.385\,535\,568 \times 60^5$	$0.299\,792\,458 \times 10^9$	$0.279\,203\,484 \times 2^{30}$	$\text{m} \cdot \text{s}^{-1}$
\mathbb{G}	$0.000\,652\,187 \times 60^{-5}$	$0.000\,838\,717 \times 10^{-95}$	$0.000\,225\,141 \times 2^{-285}$	$\text{g}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2}$
$F \bullet q$	5.331 925 229	5.290 817 692	4.765 544 917	mC
$^\circ \bullet a$	0.393 528	0.399 667	0.492 628	K
$q \bullet m$	79.569 318	61.396 079	103.005 528	g
$f \bullet l$	0.349 131	0.572 947	0.297 491	m
$h \bullet t$	0.905 574	1.911 147	1.065 500	s
$^\circ \bullet \bullet a$	0.356 369 379	0.763 823 258	0.524 895 346	K
$q \bullet \bullet m$	72.055 936 326	117.336 939 202	109.752 402 408	g
$f \bullet \bullet l$	0.385 535 568	0.299 792 458	0.279 203 484	m
$^\circ \bullet \bullet \bullet a$	0.890 923 448	0.884 054 697	0.398 142 840	K
$q \bullet \bullet \bullet m$	180.139 840 815	135.806 642 595	83.249 229 678	g
$f \bullet \bullet \bullet l$	0.154 214 227	0.259 020 684	0.368 090 530	m
$h \bullet \bullet \bullet t$	0.400 000 000	0.864 000 000	1.318 359 375	s
	vicenary ϕ	duodenary \mathfrak{d}	ternary θ	
k	$9.048\,221\,286 \times 20^{-16}$	$52.930\,768\,998 \times 12^{-20}$	$13.596\,229\,613 \times 3^{-44}$	$\text{K}^{-1} \cdot \text{g} \cdot \text{m}^2 \cdot \text{s}^{-2}$
h	$1.769\,277\,917 \times 20^{-24}$	$25.033\,037\,071 \times 12^{-30}$	$9.776\,878\,706 \times 3^{-67}$	$\text{g} \cdot \text{m}^2 \cdot \text{s}^{-1}$
c	$0.234\,212\,858 \times 20^7$	$0.697\,221\,442 \times 12^8$	$0.257\,938\,912 \times 3^{19}$	$\text{m} \cdot \text{s}^{-1}$
\mathbb{G}	$0.001\,073\,558 \times 20^{-7}$	$0.004\,327\,597 \times 12^{-9}$	$0.000\,108\,312 \times 3^{-17}$	$\text{g}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2}$
$F \bullet q$	2.167 118 927	8.151 574 232	5.094 303 677	mC
$^\circ \bullet a$	0.119 110	0.583 246	0.746 681	K
$q \bullet m$	19.646 745	63.506 581	152.587 896	g
$f \bullet l$	0.384 498	0.565 358	0.248 406	m
$h \bullet t$	1.641 662	0.810 873	0.963 044	s
$^\circ \bullet \bullet a$	0.195 538 754	0.472 939 229	0.719 087 496	K
$q \bullet \bullet m$	32.253 332 255	51.495 831 304	146.948 954 583	g
$f \bullet \bullet l$	0.234 212 858	0.697 221 442	0.257 938 912	m
$^\circ \bullet \bullet \bullet a$	0.362 108 804	1.362 064 979	0.491 451 360	K
$q \bullet \bullet \bullet m$	59.728 393 065	148.307 994 155	100.430 426 148	g
$f \bullet \bullet \bullet l$	0.126 474 943	0.242 090 778	0.377 414 046	m
$h \bullet \bullet \bullet t$	0.540 000 000	0.347 222 222	1.463 191 587	s