Handicapping for Club Sailors

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From ***The Racing Rules of Sailing***

RRS Definition for ***Rule*** alternative (d)

> *the class rules (for a boat racing under a handicap or rating system, the rules of that system are 'class rules');*

RRS Rule 78.1 — COMPLIANCE WITH CLASS RULES; CERTIFICATES

> *… shall ensure that the boat is maintained to comply with her class rules and that her measurement or rating certificate, if any, remains valid. …*

RRS Rule A3 — STARTING TIMES AND FINISHING PLACES

> *The time of a boat's starting signal shall be her starting time, and the order in which boats finish a race shall determine their finishing places. However, when a handicap or rating system is used a boat's corrected time shall determine her finishing place.*

RRS Rule A7 — RACE TIES

> *If boats are tied at the finishing line or if a handicap or rating system is used and boats have equal corrected times, …*

And so we exhaust the rule book's mentioning of handicapping, other than the *Notice of Race* and *Sailing Instruction* guides. It is left to the relevant class associations to define what the terms mean and how they are applied — which is a great pity as almost all handicapping systems are of a kind. We will address these commonalities and explore the extremely promising two-factor handicapping.

At the club level, many fleets effectively define their own class association based on the published handicaps and documentation of an existing handicap or rating system. These documents usually get caught up in the mechanics and minutiae of its own particular methods and miss out on the bigger picture. We can do better.

In particular, we will explore the arithmetic of handicapping. The reckoning of corrected times and time allowances is not difficult and does not require sophisticated mathematics to be understood. But published materials have been so misguided and so many half-truths have been perpetuated that a thorough and correct treatment is needed.

Contents

Reading the Book in Parts

The separate parts of the book were melded into a whole after the fact so there is a fair amount of repetition. This is bad but also good; all the parts are united in theme but each can be read in isolation.

Target Audience by Chapter

This book contains a fair amount of mathematics and quite a lot of mathematical symbols. The symbol to text ratio favours the more mathematically literate.

Chapters of Part I — Preamble

This first part contains content that is relevant throughout the book. There is some algebra but it should accessible to any audience with primary school math.

1 Definitions and Conventions Used Throughout the Book

This should be read before later sections in the book.

2 Excursus on Mental Arithmetic and Time Allowances

It would be worthwhile to read this self-contained chapter now and reread it later. It is informed by Part III — The Arithmetic of Applying Handicaps but does not rely upon it — indeed it provides context for better understanding that part of the book. We cover what a competitor needs to know while on the water by elaborating on how to understand the proportions inherent to time-on-distance and time-on-time handicapping and how to use them.

Chapters of Part II — Two-Factor Handicapping Executive Summary

The first part is an external document that was brought into the book as an introduction and to provide reasons for why the book was written. As such it is quite opinionated and not nearly as general as the following parts of the book. This part sacrifices comprehensive coverage for a more focused and yet more accessible overview.

3 Handicapping Today and the Wider Context

Not an equation in sight. This chapter is for a general audience and describes the problems to be solved. We don't need to understand how a velocity prediction programme works, but we do need to know how it should be applied to sailboat racing — there are no good references. This book seeks to address that deficit and this chapter is a good introduction on why this book is needed.

4 Time-on-Time-and-Distance Exposition

Here we introduce performance predictions via simple linear relationships, determine time allowances from them and finally compute corrected times. But the overall effect is not excessively mathematical.

5 A Critique of Handicapping

A brief but more sophisticated mathematical extension of the previous arguments informs the critique that follows.

Chapters of Part III — The Arithmetic of Applying Handicaps

This part is intended for a general audience with an advanced high-school level of algebra; although, anyone with basic high-school level proficiency should appreciate the first few chapters. Note this part of the book is independent of Part II, the executive summary, and has a different structure to it. That part was more organic and developed handicapping and corrected times in a natural order, justifying the algebra as it was introduced. This part, Part III, is more comprehensive and introduces the formalisms as would a textbook, axiomatically without prior justification and then fleshing out the details as quickly as possible.

6 Introduction

Conventions used throughout this part and, to a lesser degree, throughout the book.

7 On Distance, On Time or On Time and Distance

This chapter should be accessible to most readers. It covers most everything a competitor needs to know. It presupposes a certain familiarity with equations, simple linear relationships, basic units of space and time, ratios and proportions, graphs, basic parametric equations and functions and not panicking when seeing a free variable.

8 A General Purpose Handicap

This chapter is an adjunct to the previous chapter, using the knowledge built to better compare the single-factor styles of handicapping.

9 Interpreting Intervals of Corrected Time

The simplest chapter of all and a must-read for all sailors trying to interpret corrected times in published race results.

10 Absolute versus Relative Performance

This chapter is of interest to competitors who have to deal with old-fashioned relative-gauge handicaps; unfortunately, that is most of us. Mathematically, this is pretty straightforward. Gauge conversions and transformations arise naturally in this context but are a bit more abstract.

11 Programming Corrected Times Without Rounding

This chapter is for computer programmers writing scoring software and of little interest to anyone else.

12 Positive-Sense versus Negative-Sense Handicaps

The organizers of pursuit races need penalizing positive-sense handicaps but everyone else is better served by negative-sense corrective handicaps. Converting from one style to the other is remarkably simple algebraically, and if only handicapping authorities would do the right thing this chapter would be a lot shorter.

13 Handicapping 2×2 Matrix Notation

This sections is of interest to handicapping authorities wanting to arithmetically manipulate time-on-time-and-distance handicaps.

14 Standard Units and Variables in those Units

This reference chapter has tables of variables in the two different unit schemes (gauges of *preserved* or *flattened* dimensionality).

Chapters of Part IV — Computing Performance Handicaps in Bulk

This part is for more mathematically sophisticated readers, with at least a first-year university level of algebra; although, it doesn't actually contain much new or unexpected.

15 General Concerns

A mostly textual overview of the mathematical problem and outlining solutions. We also introduce variable conventions used throughout this part of the book.

16 Using Least Squares to Compute Handicaps

Least squares is the technique used to identify, out of all possible handicaps, those which are the *best*. It's not a complicated criterion to understand, being closely related to the arithmetic mean, and is extremely well studied. We look into solutions that satisfy additional *gauge criteria* to yield unique solutions whenever possible. We also introduce the *log transform* used to linearize time-on-time handicapping.

17 Time-on-Distance As a Linear Model

Time-on-distance and time-on-time via a log transform are the only styles of handicapping to directly support solutions which can be expressed linearly in terms of the inputs. *Linear regression* and the *analysis of variance* provides a solution to this handicapping problem. We will recapitulate the analysis in the main.

18 Solvers for Nonlinear Models

Iterative solvers use an initial guess for the control, test it against the criteria for optimality, and then refine the guess using the local derivatives to estimate the behaviour of the whole. A very simple but quickly computed iterative step may lead to a solution in less time than a more sophisticated solver using fewer iterations.

Chapters of Part V — Running Statistics for Performance Handicaps

This part is for computing handicaps interactively, either year-by-year or race-by-race.

19 Baysian Statistics through Monte Carlo Simulations

Baysian statistics is the best, arguably the only, theoretical framework which describes the statistical methods appropriate for computing updates to performance handicaps as racing progresses. Baysian methods hardly ever yield simple analytic results which makes the whole field difficult to approach. Monte Carlo simulations seem quite ridiculous at first sight but work just as well as iterative solvers.

20 Recursive Least Squares

Means and covariance matrices are often sufficient to encapsulate all the historical information necessary for determining handicaps in an ongoing fashion. Recursive least squares ...

21 Kalman Filters

Kalman filters are simple linear methods ...

Part I

Preamble

Chapter 1

Definitions and Conventions Used Throughout the Book

1.1 A Rating versus a Handicap

It is worth mentioning the traditional difference between a *rating* and a *handicap*. Handicaps were a measure of relative performance to be applied directly to a corrected time formula and, as such, were either unitless fractions close to a power of ten (as a time coefficient) or in units of seconds per nautical mile (as a distance coefficient). The traditional *rating* was a measure of length, waterline length being the most significant predictor of performance in the classic yacht. It was calculated by a complicated empirical formula combining length, sail area, displacement and other measurements into a bastardized whole. A handicap could then be derived from the rating via an another empirical formula.

In modern usage *rating* and *handicap* are synonymous. Both are predictions of absolute performance (sometimes implicitly) and best expressed in units of seconds per nautical mile.

1.2 Polars and Variants

The traditional *rated length* was a single number and, as such, couldn't represent the difference in performance as the wind varied or across different points of sail. A *velocity prediction programme* (VPP) will find a boat's optimal pace in seconds per nautical mile across a sampling of points of sail and wind speeds. Putting these together gives the *polars* (or the *polar diagrammes*) for the boat. Variations in the sail plan or other factors which affect performance may also be included in the model to create a boat's overall performance profile. These may then be integrated before a race to get the predicted performance suitable for a race or series or, more likely for club racing, be integrated for the predicted performance on a *typical* course for a *normally* configured boat.

Planing boats introduce discontinuities in the VPP which makes prediction much harder — no rule exists which can adequately handicap between planing and nonplaning boats — the best that we can be hope for is to average out the expected performance differences over a series.

1.3 Specialization

With certificates in hand, many handicap or rating rules still allow for event specific handicapping. Portsmouth handicaps come in light, medium and heavy air variants. ORR and ORC provide *offshore* or *round-the-buoys* course variants to match the expected points of sail throughout the race. A more precise predicted pace can be integrated from full polars and the expected winds and currents on a course.

Performance curve scoring is an innovation introduced by IMS and still available for use. It is a clever way to integrate the polars around a course (on the predicted points of sail) across a wide range of wind speeds before racing and then, once racing in underway, to fit the observed pace to the resultant *performance curve* to objectively respond to changes in the wind; a fitting which can be applied by competitors throughout the race as well as by the Race Committee at the finish line. Informing competitors of a full performance curve requires computerization or pre-computed time allowance tables.

More specifically, performance curve scoring as originally defined can be applied by competitors throughout a race. The bastardized version as now advertized by ORC breaks the promise of objectivity by making everyone's handicap depend on the leading boat's wind speed as imputed from its elapsed time, an observation not necessarily available to all competitors. And even when the elapsed time of the leader is apparent, this is still horrible for competitors. Consider two boats close to each other both on the race course and as rated — that they should be gauged by the arbitrary performance of a distant division leader is madness. The ORC rules document suggest that Race Committees should input a wind speed when the implied wind speed of the leader isn't representative of the fleet making clear this is a parametric handicap akin to the light, medium and heavy air variants but where the parameter is unknowable until after the race is over. This broken implementation of performance curve scoring should never be used — even the association by name with true performance curve scoring is best avoided.

Two-factor time-on-time-and-distance handicapping (sometimes called *performance line scoring*) is a variation on performance curve scoring which trades precision for simplicity while retaining objectivity. Time-on-time-and-distance handicaps are easy to apply both by competitors and the Race Committee yet still yield predictions which are implicitly responsive to changes in wind speed.

Club races use very objective handicapping for the lack of the race organization needed to apply subjective fine-tuning (for this reason IRC and PHRF offer no opportunity to do so). Within a series courses can be held to a standard but winds can not. The cost in precision of using an unvarying single-factor handicap is such that individual club races can be quite unfair, with handicaps averaging out in predictive power only over a series of races. In this regard, time-on-time-and-distance handicapping appears to be ideal for club racing.

1.4 Common Units and Variable Conventions

Let's state some common variable name conventions. We will use d for *course length* in nautical miles (we will always read the abbreviation *mi.* as *nautical mile*). *Elapsed times* are the duration in time from the starting signal. We will use t and $\check{\check{t}}$ (with two *checks* on top — read *check-check*) for elapsed and corrected times in seconds. Dividing through by course length gives us the *course-average pace* $p = t \div d$ and the *corrected pace* (*corrected course-average pace*) $\check{\check{p}} = \check{\check{t}} \div d$ in seconds per nautical mile. We can think of these as the elapsed and corrected times on a course normalized in length to one nautical mile.

Pace is a measure of how long it takes to complete a distance and varies inversely to speed v measured in knots. To convert between pace and speed

$$v = \frac{3600\,{}^{\mathrm{s}}\!/\!_{\mathrm{hr}}}{p} \qquad p = \frac{3600\,{}^{\mathrm{s}}\!/\!_{\mathrm{hr}}}{v}$$

So, for example, a speed of $v = 6\,\mathrm{kt}$ corresponds to a pace of $p = 600\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}}$ and a speed of $v = 4\,\mathrm{kt}$ corresponds to a pace of $p = 900\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}}$. Note that a slower pace is represented by a greater number of seconds per mile. Pace is the natural measure of performance prediction and handicapping. Unqualified, *speed* more often means *instantaneous speed* which can vary continuously with time rather than *average speed* which is measured over a distance. Whereas unqualified *pace* more often means *average pace*. How we are using these terms should be obvious in context.

We will be using single letter variable names for the most part — annotated with various subscripts and superscripts as the situation demands. An exception will be variables to represent a difference using the capital Greek letter Δ (*delta*) as a variable prefix. So Δt (*delta-t*) will be a difference in elapsed times and $\Delta\check{\check{t}}$ (read *delta-t-check-check*) a difference in corrected time. We will often employ a convention where differences are relative to a fixed value denoted with a star, a prime, a circle or suchlike $\Delta x \equiv x - {}^{\bigstar}x$. For two boats we may refer to a left and right boat with the right boat identified with a prime so that $\Delta x \equiv x - x'$.

Putting a hat on an unknown variable x to get $\hat{x}$ (read *x-hat*) denotes a prediction of some kind. A handicap is essentially a prediction of pace $q \mapsto \hat{p}$ with a single degree of freedom, here denoted q for a generalized co-ordinate in no particular units, which encapsulates how fast or slow a particular race will be. This prediction operates inversely to applying a correction to an observed pace $p \mapsto \check{q}$. That *check* looks like an upside-down *hat* serves to emphasize this connection. A corrected pace is the result of composing two operations, a simple correction $p \mapsto \check{q}$ followed by a prediction $\check{q} \mapsto \check{\check{p}}$ appropriate for a chosen scratch boat $\bigstar$ — this could be represented symbolically by annotating the time or pace variable with a single *check* and then *star-hat* stacked atop that ($\check{\hat{\star}}$) but that would be a very unwieldy notation to carry around — two *checks* serves us better. Note that the composite action of the two opposing operations inherent to computing corrected time or pace ensure that $\check{\check{t}}$ and $\check{\check{p}}$ are measured in the same units of time and pace respectively. This is convenient because we almost always want the units for a variable to determined by its *letter* form and its spoken name rather than by any decorations applied to the variable.

Time allowances aren't predictions but critical intervals of elapsed time necessary to secure a placing after handicapping has been applied. The mechanisms of handicapping prediction are used to determine these critical times so both the Δ notation for differences of time and the *hat* notation for predictions will be relevant in exploring how to understand and use time allowances to best effect.

The form for the variable will be suggestive but not fully define its usage; for that we will rely on the accompanying text. In this we differ from many texts which prefer longer variable names composed of initials and also from computer code which generally prefers longer descriptive names. The short variables names lead to compact algebraic expressions which are easy to read. Also we will be cover a lot of material from slightly different viewpoints, using the same or similar variable names for related concepts will make the material more approachable than the big-bag-of-acronyms style would allow.

Note that we will be overloading the $\bigstar$ notation to identify the scratch boat in corrected time calculations, your own boat in time allowances and the standard boat in relative gauge handicapping manipulations and a few more to boot — in this exposition we are trying to stress the similarities of these calculations but in any other context we would usually choose distinct notational conventions to better identify the context — in related documents we will use a hollow circle $\circ$ for your own boat, will keep the big star for the scratch boat within your division and will use a small star $\star$ for the handicapping authority's standard boat.

Chapter 2

Excursus on Mental Arithmetic and Time Allowances

This excursus can be read before or after other sections or can be skipped entirely. It is written with respect to time-on-distance and time-on-time handicapping.

2.1 A General Purpose Handicap either On Time or On Distance

2.1.1 Pace and Handicaps

A general purpose handicap g is a boat's pace on average and can be used for either time-on-distance or time-on-time handicapping. A slower pace is represented by a greater number of seconds per mile and the following example boats are ordered from fastest to slowest. The "delta-gee" Δg column shows the differences in handicap from our own boat, *Shindig*, which we identify with a star ★.

Example Boat	g	Δg	Make
Hurricane	729(12:09)	−132(2:12)	Buddy 24
Winged Elephant	810(13:30)	−51	Frequency 24
Mechanical Drone	834(13:54)	−27	See in Sea 30
Shindig	861(14:21)	★	Raider 28
Professor	864(14:24)	+3	Stone 22
Rhumb Punch	876(14:36)	+15	Chimera 33

Units are not shown in the table but are understood to be seconds per mile with the equivalent minutes and seconds per mile in brackets. These handicaps are rounded to the closest multiple of $3\,\mathrm{s/mile}$ so it will be natural to reckon time allowances in unit thirds.

A PHRF rating (or any time-on-distance rating system with handicaps in units of seconds per mile) is the difference in general purpose handicap from that of the zero-rated boat. Should the zero-rated boat have as its general purpose handicap ${}^{\text{zero-rated}}g = 600\,\mathrm{s/mile}$ (for example) then adding $600\,\mathrm{s/mile}$ to our own boat's PHRF rating will recover our boat's general purpose handicap. Further note that a difference in general purpose handicaps is equal to the corresponding difference in PHRF ratings

$$\Delta g = \Delta\mathrm{PHRF}$$

It turns out that all we need to compare ourselves to our competitors on the water is our own boat's general purpose handicap together with the "delta" in PHRF ratings.

2.1.2 A Time Allowance from Our Perspective

A time allowance “delta-tee” Δt is the time ahead or behind us that a competing boat must finish in order to tie with us after handicapping is applied. Likewise a pace allowance “delta-pee” Δp is a difference in course-average pace necessary for a tie. Multiplication by course length d connects our course-average pace p to our elapsed time t and a pace allowance Δp to a time allowance Δt.

For time-on-distance handicapping the pace allowance for a competitor Δp is fixed at Δg. By multiplying with the known course length, the time allowance is predetermined and wont vary however long it takes us to finish the course.

For time-on-time handicapping the relationship between our observed course-average pace p and the pace allowance Δp is best expressed as a proportionality. The ratio of Δp to Δg is equal in proportion to the ratio of p to g. Time-on-time handicapping is independent of course length so turning a pace allowance into a time allowance can be achieved by simply dropping per-mile from all the units in the proportionality. We’ll show this more thoroughly in the worked examples below.

2.1.3 Our Boat *Shindig*

Our handicap is $g = 861\,\mathrm{s/mile} = 14\,\mathrm{min}\,21\,\mathrm{s/mile}$. On average Shindig should take $861\,\mathrm{s} = 14\,\mathrm{min}\,21\,\mathrm{s}$ to complete a mile of the course or, in thirds, $287\,\mathrm{s} = 4\,\mathrm{min}\,47\,\mathrm{s}$ to complete a third of a mile. If a race course were four and one-third miles long we would add the expected elapsed time on a four mile course to that on a third of a mile course. Using the “varies in proportion to” $\propto$ notation

$$
\begin{array}{rrl}
 & 57\,\mathrm{min}\,24\,\mathrm{s} \propto & 4\,\mathrm{mile}\\
+ & 4\,\mathrm{min}\,47\,\mathrm{s} \propto & 1/3\,\mathrm{mile}\\ \hline
 & 62\,\mathrm{min}\,11\,\mathrm{s} \propto & 4\,1/3\,\mathrm{mile}
\end{array}
\quad
\left\{
\begin{array}{rll}
t \propto & d & \text{(on average)}\\
14\,\mathrm{min}\,21\,\mathrm{s} \propto & 1\,\mathrm{mile} & \\ \hline
28\,\mathrm{min}\,42\,\mathrm{s} \propto & 2\,\mathrm{mile} & (2\times)\\
43\,\mathrm{min}\,\;\,3\,\mathrm{s} \propto & 3\,\mathrm{mile} & (3\times)\\
57\,\mathrm{min}\,24\,\mathrm{s} \propto & 4\,\mathrm{mile} & (4\times)\\
1\,\mathrm{h}\,11\,\mathrm{min}\,45\,\mathrm{s} \propto & 5\,\mathrm{mile} & (5\times)\\
 & \vdots & \\ \hline
4\,\mathrm{min}\,47\,\mathrm{s} \propto & 1/3\,\mathrm{mile} & (1/3\times)\\
9\,\mathrm{min}\,34\,\mathrm{s} \propto & 2/3\,\mathrm{mile} & (2/3\times)
\end{array}
\right.
$$

Were we to finish this course with an elapsed time of $1\,\mathrm{h}\,2\,\mathrm{min}\,11\,\mathrm{s}$ then all the time allowances calculated with regard to time-on-time would be the same as for time-on-distance.

We demonstrated techniques of mental arithmetic in the working above. For example, we replaced a multiplication with sequential additions. Getting from $14\,\mathrm{min}\,21\,\mathrm{s}$ to $28\,\mathrm{min}\,42\,\mathrm{s}$ was easy, we simply doubled all the digits. To get to the multiple of three we add these. $28\,\mathrm{min}$ plus $14\,\mathrm{min}$ is $42\,\mathrm{min}$. $42\,\mathrm{s}$ plus $21\,\mathrm{s}$ is $63\,\mathrm{s}$ giving us $42\,\mathrm{min}\,63\,\mathrm{s}$ in all. The seconds overflowed so we reshuffle this to get $43\,\mathrm{min}\,3\,\mathrm{s}$. We always works from the big to the little (bigendian!) — we start with the largest effect and then refine it with further smaller adjustments to improve the accuracy of the result. If we can work out the largest effects beforehand and write them in a table we can save ourselves a lot of work — time allowances are well suited this. At the very least we should always have a table of differences in handicap and, if using time-on-time, a table of expected elapsed times at reasonable intervals.

2.2 Reckoning Time Allowances from Our Boat *Shindig's* Point of View

2.2.1 Between Us and Our Competitor *Rhumb Punch*

The table of handicaps states that $\Delta g = 15\,\mathrm{s/mile}$ (Or equivalently that $\Delta\mathrm{PHRF} = 15\,\mathrm{s/mile}$). For time-on-distance handicapping every mile of course length contributes 15 s to the time allowance Δt. For each additional 1/3 mile the time allowance is increased by 5 s. On a four and one-third mile course this would yield a time allowance of 65 s

$$
\begin{array}{rrr}
 & 60\,\mathrm{s} \propto & 4\,\mathrm{mile} \\
+ & 5\,\mathrm{s} \propto & 1/3\,\mathrm{mile} \\
\hline
 & 65\,\mathrm{s} \propto & 4\,1/3\,\mathrm{mile}
\end{array}
\quad
\left\{
\begin{array}{rll}
\Delta t \propto & d\ \text{(time-on-distance)} & \\
15\,\mathrm{s} \propto & 1\,\mathrm{mile} & \\
\hline
30\,\mathrm{s} \propto & 2\,\mathrm{mile} & (2\times) \\
45\,\mathrm{s} \propto & 3\,\mathrm{mile} & (3\times) \\
60\,\mathrm{s} \propto & 4\,\mathrm{mile} & (4\times) \\
\hline
5\,\mathrm{s} \propto & 1/3\,\mathrm{mile} & (1/3\times)
\end{array}
\right.
$$

For time-on-time handicapping the ratio of the time allowance Δt to 15 s is equal in proportion to the ratio of elapsed time t to 14 min 21 s. That is, for every 14 min 21 s of elapsed time t the time allowance Δt increases by 15 s. In unit thirds, for every 4 min 47 s of elapsed time the time allowance increases by 5 s. At an elapsed time of 1 h 2 min 11 s we would expect a 65 s time allowance, the same as for time-on-distance handicapping on a four and one-third mile course

$$
\begin{array}{rrr}
 & 60\,\mathrm{s} \propto & 57\,\mathrm{min}\,24\,\mathrm{s} \\
+ & 5\,\mathrm{s} \propto & 4\,\mathrm{min}\,47\,\mathrm{s} \\
\hline
 & 65\,\mathrm{s} \propto & 62\,\mathrm{min}\,11\,\mathrm{s}
\end{array}
\quad
\left\{
\begin{array}{rll}
\Delta t \propto & t\ \text{(time-on-time)} & \\
15\,\mathrm{s} \propto & 14\,\mathrm{min}\,21\,\mathrm{s} & \\
\hline
30\,\mathrm{s} \propto & 28\,\mathrm{min}\,42\,\mathrm{s} & (2\times) \\
45\,\mathrm{s} \propto & 43\,\mathrm{min}\ \ 3\,\mathrm{s} & (3\times) \\
60\,\mathrm{s} \propto & 57\,\mathrm{min}\,24\,\mathrm{s} & (4\times) \\
\hline
5\,\mathrm{s} \propto & 4\,\mathrm{min}\,47\,\mathrm{s} & (1/3\times)
\end{array}
\right.
$$

The ratio of 65 s to 15 s is equal in proportion to the ratio of 62 min 11 s to 14 min 21 s. To reiterate: the ratio of the reckoned time allowance to the difference in handicaps (dropping per-mile from the unit) is equal in proportion to the ratio of our own elapsed time to our own general purpose handicap (dropping per-mile from the unit).

The overall pattern is obvious. On average Δt, t and d vary in lockstep

$$
\begin{array}{rrrr}
 & 60\,\mathrm{s} \propto & 57\,\mathrm{min}\,24\,\mathrm{s} \propto & 4\,\mathrm{mile} \\
+ & 5\,\mathrm{s} \propto & 4\,\mathrm{min}\,47\,\mathrm{s} \propto & 1/3\,\mathrm{mile} \\
\hline
 & 65\,\mathrm{s} \propto & 62\,\mathrm{min}\,11\,\mathrm{s} \propto & 4\,1/3\,\mathrm{mile}
\end{array}
\quad
\left\{
\begin{array}{rrll}
\Delta t \propto & t \quad \propto & d & \\
15\,\mathrm{s} \propto & 14\,\mathrm{min}\,21\,\mathrm{s} \propto & 1\,\mathrm{mile} & \\
\hline
30\,\mathrm{s} \propto & 28\,\mathrm{min}\,42\,\mathrm{s} \propto & 2\,\mathrm{mile} & (2\times) \\
45\,\mathrm{s} \propto & 43\,\mathrm{min}\ \ 3\,\mathrm{s} \propto & 3\,\mathrm{mile} & (3\times) \\
60\,\mathrm{s} \propto & 57\,\mathrm{min}\,24\,\mathrm{s} \propto & 4\,\mathrm{mile} & (4\times) \\
\hline
5\,\mathrm{s} \propto & 4\,\mathrm{min}\,47\,\mathrm{s} \propto & 1/3\,\mathrm{mile} & (1/3\times)
\end{array}
\right.
$$

For an actual race which departs from the average, time allowances are dependent on either time or distance depending on the style of handicapping. Were we to take exactly one hour to finish a race using time-on-time handicapping, the time allowance for 57 min 24 s would fall short and the time allowance for 62 min 11 s would overshoot. But we only need about two and half minutes worth of additional time allowance to round out the 57 min 24 s worth. As a rough estimate every five minutes of elapsed time increases the time allowance by five seconds. So $2.5\,\mathrm{s} \propto 2.5\,\mathrm{min}$, approximately. This would give a total time allowance of about 62.5 s. To be certain of the win, we must cross the finish line at least 1 min 3 s before Rhumb Punch.

2.2.2 Between Us and Our Competitor *Professor*

We have $\Delta g = 3\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mile}}$ from the table of handicaps. On the same four and one-third mile course with time-on-distance handicapping or taking the same 62 min 11 s with time-on-time handicapping

$$
\begin{array}{rrcrcr}
 & 12\,\mathrm{s} & \propto & 57\,\mathrm{min}\,24\,\mathrm{s} & \propto & 4\,\mathrm{mile} \\
+ & 1\,\mathrm{s} & \propto & 4\,\mathrm{min}\,47\,\mathrm{s} & \propto & 1/3\,\mathrm{mile} \\
\hline
 & 13\,\mathrm{s} & \propto & 62\,\mathrm{min}\,11\,\mathrm{s} & \propto & 4\,1/3\,\mathrm{mile}
\end{array}
\quad
\left\{
\begin{array}{rcrcr}
\Delta t & \propto & t & \propto & d \\
3\,\mathrm{s} & \propto & 14\,\mathrm{min}\,21\,\mathrm{s} & \propto & 1\,\mathrm{mile} \\
\hline
6\,\mathrm{s} & \propto & 28\,\mathrm{min}\,42\,\mathrm{s} & \propto & 2\,\mathrm{mile} \\
9\,\mathrm{s} & \propto & 43\,\mathrm{min}\,\ \,3\,\mathrm{s} & \propto & 3\,\mathrm{mile} \\
12\,\mathrm{s} & \propto & 57\,\mathrm{min}\,24\,\mathrm{s} & \propto & 4\,\mathrm{mile} \\
\hline
1\,\mathrm{s} & \propto & 4\,\mathrm{min}\,47\,\mathrm{s} & \propto & 1/3\,\mathrm{mile}
\end{array}
\right.
$$

For every mile of distance or for every 14 min 21 s of elapsed time, the time allowance we must give the Professor increases by 3 s. Likewise, For every third of a mile or 4 min 47 s the time allowance increases by 1 s. We can repeat this with the Δg for each of our competitors to describe all the time allowances we need.

2.2.3 For the Competitors we have seen So Far

Adding the superscript $^{\mathrm{Prof}}$ for the Professor and $^{\mathrm{RP}}$ for Rhumb Punch

$$
\begin{array}{rrcrcrcr}
 & 12\,\mathrm{s} & \propto & 60\,\mathrm{s} & \propto & 57\,\mathrm{min}\,24\,\mathrm{s} & \propto & 4\,\mathrm{mile} \\
+ & 1\,\mathrm{s} & \propto & 5\,\mathrm{s} & \propto & 4\,\mathrm{min}\,47\,\mathrm{s} & \propto & 1/3\,\mathrm{mile} \\
\hline
 & 13\,\mathrm{s} & \propto & 65\,\mathrm{s} & \propto & 62\,\mathrm{min}\,11\,\mathrm{s} & \propto & 4\,1/3\,\mathrm{mile}
\end{array}
\quad
\left\{
\begin{array}{rcrcrcr}
\overset{\mathrm{Prof}}{\Delta t} & \propto & \overset{\mathrm{RP}}{\Delta t} & \propto & t & \propto & d \\
3\,\mathrm{s} & \propto & 15\,\mathrm{s} & \propto & 14\,\mathrm{min}\,21\,\mathrm{s} & \propto & 1\,\mathrm{mile} \\
\hline
6\,\mathrm{s} & \propto & 30\,\mathrm{s} & \propto & 28\,\mathrm{min}\,42\,\mathrm{s} & \propto & 2\,\mathrm{mile} \\
9\,\mathrm{s} & \propto & 45\,\mathrm{s} & \propto & 43\,\mathrm{min}\,\ \,3\,\mathrm{s} & \propto & 3\,\mathrm{mile} \\
12\,\mathrm{s} & \propto & 60\,\mathrm{s} & \propto & 57\,\mathrm{min}\,24\,\mathrm{s} & \propto & 4\,\mathrm{mile} \\
\hline
1\,\mathrm{s} & \propto & 5\,\mathrm{s} & \propto & 4\,\mathrm{min}\,47\,\mathrm{s} & \propto & 1/3\,\mathrm{mile}
\end{array}
\right.
$$

2.2.4 To Summarize

For each of our competitors Δg or ΔPHRF ($\times 1$ mile) is the difference in handicap dropping per-mile from the unit: in the time-on-distance case each mile of the course contributes this to the time allowance for the corresponding boat; whereas in the time-on-time case each 14 min 21 s of our own elapsed time contributes this to the time allowance. Here 14 min 21 s is just our own general purpose handicap g dropping per-mile from the unit ($\times 1$ mile).

2.2.5 The Table of Handicaps Also Gives Us

The table of handicaps also gives us $\overset{\mathrm{MD}}{\Delta g} = -27\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mile}}$, $\overset{\mathrm{WE}}{\Delta g} = -51\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mile}}$ and $\overset{\mathrm{Hurr}}{\Delta g} = -132\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mile}}$ for our competitors *Mechanical Drone*, *Winged Elephant* and *Hurricane* respectively. The negative sign simply means the time allowance is in our favour — we will drop the sign (with a little finesse) in the presentation below. When expressing variations in proportion over multiple boats it is more conventional to write the distance and time on the left and the have per-boat time allowances on the right, where we order competitors by the magnitude of Δg (and using superscripts on the variables to identify competitors)

$$
\begin{array}{rcrcrcrcrcrcr}
d & \propto & t & \propto & \overset{\text{Prof}}{\Delta t} & \propto & \overset{\text{RP}}{\Delta t} & \propto & -\overset{\text{MD}}{\Delta t} & \propto & -\overset{\text{WE}}{\Delta t} & \propto & -\overset{\text{Hurr}}{\Delta t} \\
1\,\text{mile} & \propto & 14\,\text{min}\,21\,\text{s} & \propto & 3\,\text{s} & \propto & 15\,\text{s} & \propto & 27\,\text{s} & \propto & 51\,\text{s} & \propto & 132\,\text{s} \\
\hline
2\,\text{mile} & \propto & 28\,\text{min}\,42\,\text{s} & \propto & 6\,\text{s} & \propto & 30\,\text{s} & \propto & 54\,\text{s} & \propto & 102\,\text{s} & \propto & 264\,\text{s} \\
3\,\text{mile} & \propto & 43\,\text{min}\ \ 3\,\text{s} & \propto & 9\,\text{s} & \propto & 45\,\text{s} & \propto & 81\,\text{s} & \propto & 153\,\text{s} & \propto & 396\,\text{s} \\
4\,\text{mile} & \propto & 57\,\text{min}\,24\,\text{s} & \propto & 12\,\text{s} & \propto & 60\,\text{s} & \propto & 108\,\text{s} & \propto & 204\,\text{s} & \propto & 528\,\text{s} \\
\hline
1/3\,\text{mile} & \propto & 4\,\text{min}\,47\,\text{s} & \propto & 1\,\text{s} & \propto & 5\,\text{s} & \propto & 9\,\text{s} & \propto & 17\,\text{s} & \propto & 44\,\text{s}
\end{array}
$$

This presentation is mathematically precise but visually cluttered. Proportions are highly suited to being expressed in a table; whereas, the above notation is best suited for annotating additions.

$$
\begin{array}{crcrcrcrcrcrcr}
 & 4\,\text{mile} & \propto & 57\,\text{min}\,24\,\text{s} & \propto & 12\,\text{s} & \propto & 60\,\text{s} & \propto & 108\,\text{s} & \propto & 204\,\text{s} & \propto & 528\,\text{s} \\
+ & 1/3\,\text{mile} & \propto & 4\,\text{min}\,47\,\text{s} & \propto & 1\,\text{s} & \propto & 5\,\text{s} & \propto & 9\,\text{s} & \propto & 17\,\text{s} & \propto & 44\,\text{s} \\
\hline
 & 4\,1/3\,\text{mile} & \propto & 62\,\text{min}\,11\,\text{s} & \propto & 13\,\text{s} & \propto & 65\,\text{s} & \propto & 117\,\text{s} & \propto & 221\,\text{s} & \propto & 572\,\text{s}
\end{array}
$$

2.2.6 Expressing Proportionalities In a Table

In tables of time allowances we cut down on visual clutter by just expressing the values, omitting units, uniformly expressing intervals of time as hours:minutes:seconds and by ordering results to make them easier to look up.

We would do well to fill out the table by adding an entry for 2/3 of a mile as well as entries for 5 through 9 miles. We further reduce clutter by breaking out a legend which is cross-referenced to abbreviations used in the column headings rather than trying to stuff this into the main table.

Shindig		+Prof	+RP	−MD	−WE	−Hurr
1/3	4:47	1	5	9	17	44
2/3	9:34	2	10	18	34	1:28
1	14:21	3	15	27	51	2:12
2	28:42	6	30	54	1:42	4:24
3	43:03	9	45	1:21	2:33	6:36
4	57:24	12	1:00	1:48	3:24	8:48
5	1:11:45	15	1:15	2:15	4:15	11:00
6	1:26:06	18	1:30	2:42	5:06	13:12
7	1:40:27	21	1:45	3:09	5:57	15:24
8	1:54:48	24	2:00	3:36	6:48	17:36
9	2:09:09	27	2:15	4:03	7:39	19:48

★	Shindig	14:21
Prof	Professor	+3
RP	Rhumb Punch	+15
MD	Mechanical Drone	−27
WE	Winged Elephant	−51
Hurr	Hurricane	−2:12

A table should be prepared prior to racing and need not be tabulated by hand; indeed, a conscientious race committee would prepare such a table for us. These tables express exact proportions, but in practice we must interpolate between lines to approximate the final time allowance.

Even better, by expanding the vertical scale a bit to fill in all the intermediate rows (using as much height as fits the space available), we have a very easy to read table, more than sufficient for any around-the-buoys race with these five competitors.

2.2.7 With Full Tables of Time Allowances

The full table can help us with interpolation by refining our proportionality — minutes $\propto$ hours:minutes translates directly to seconds $\propto$ minutes:seconds. For example with Winged Elephant we can look down the column to notice that 1 min 59 s $\propto$ 33 min 29 s from which we get the excellent approximation of 2 s $\propto$ 33.5 s. So at elapsed time of 1 h 8 min we look at the table under 4²/3 mile to get a time allowance of 3 min 58 s at 1 h 6 min 58 s. We need another minute of elapsed time which is approximately another 4 s of time allowance to give a final time allowance of 4 min 2 s.

For Mechanical Drone we can look down the table to get 1 min 3 s $\propto$ 33 min 29 s for an approximate 1 s $\propto$ 33.5 s or an even more precise 1 min 57 s $\propto$ 1 h 2 min 11 s for a better approximation 2 s $\propto$ 1 min 2 s or, best of all, 3 min $\propto$ 1 h 35 min 40 s for the approximation 3 s $\propto$ 1 min 36 s; this further simplifies to 1 s $\propto$ 32 s, more precise and simpler than our first approximation drawn from the table. At 1 h 8 min we would refine the 2 min 6 s time allowance at 1 h 6 min 58 s in the table with a further 2 s to get the very accurate time allowance of 2 min 8 s.

For Hurricane 8 min 4 s $\propto$ 52 min 37 s giving an excellent approximation of 8 s $\propto$ 52.5 s and a further refinement using a poorer approximation of 1 s $\propto$ 7 s. These aren't best rational approximations in the number-theoretic sense; but they are good-enough.

With these approximate proportionalities we can accurately work out time-on-time handicapping on the water. These approximations are also best worked out beforehand. They are easily gleaned from the fully worked out table but aren't really suited to be added to a table numerically.

Shindig		+Prof	+RP	−MD	−WE	−Hurr
1/3	4:47	1	5	9	17	44
2/3	9:34	2	10	18	34	1:28
1	14:21	3	15	27	51	2:12
1 1/3	19:08	4	20	36	1:08	2:56
1 2/3	23:55	5	25	45	1:25	3:40
2	28:42	6	30	54	1:42	4:24
2 1/3	33:29	7	35	1:03	1:59	5:08
2 2/3	38:16	8	40	1:12	2:16	5:52
3	43:03	9	45	1:21	2:33	6:36
3 1/3	47:50	10	50	1:30	2:50	7:20
3 2/3	52:37	11	55	1:39	3:07	8:04
4	57:24	12	1:00	1:48	3:24	8:48
4 1/3	1:02:11	13	1:05	1:57	3:41	9:32
4 2/3	1:06:58	14	1:10	2:06	3:58	10:16
5	1:11:45	15	1:15	2:15	4:15	11:00
5 1/3	1:16:32	16	1:20	2:24	4:32	11:44
5 2/3	1:21:19	17	1:25	2:33	4:49	12:28
6	1:26:06	18	1:30	2:42	5:06	13:12
6 1/3	1:30:53	19	1:35	2:51	5:23	13:56
6 2/3	1:35:40	20	1:40	3:00	5:40	14:40
7	1:40:27	21	1:45	3:09	5:57	15:24
7 1/3	1:45:14	22	1:50	3:18	6:14	16:08
7 2/3	1:50:01	23	1:55	3:27	6:31	16:52
8	1:54:48	24	2:00	3:36	6:48	17:36
8 1/3	1:59:35	25	2:05	3:45	7:05	18:20
8 2/3	2:04:22	26	2:10	3:54	7:22	19:04
9	2:09:09	27	2:15	4:03	7:39	19:48
9 1/3	2:13:56	28	2:20	4:12	7:56	20:32
9 2/3	2:18:43	29	2:25	4:21	8:13	21:16

Shindig	$\Delta t \propto t$ Approximations			
Professor	1 s $\propto$ 4 min 47 s	1 s $\propto$ 5 min		
Rhumb Punch	5 s $\propto$ 4 min 47 s	1 s $\propto$ 1 min (via 5 s $\propto$ 5 min)		
Mechanical Drone	9 s $\propto$ 4 min 47 s	1 s $\propto$ 32 s (via 3 s $\propto$ 1 min 36 s)		
Winged Elephant	17 s $\propto$ 4 min 47 s	4 s $\propto$ 1 min 7 s	2 s $\propto$ 33.5 s	1 s $\propto$ 17 s
Hurricane	44 s $\propto$ 4 min 47 s	11 s $\propto$ 1 min 12 s	8 s $\propto$ 52.5 s	1 s $\propto$ 7 s

2.2.8 Complementary Tables of Time Allowances

How do time allowances compare when taken from these different points of view? Unlike before, we are including the sign of the time allowance in the body of the table to highlight the complementary columns.

Boat		g	Δg	Δg	Δg	Make
Hurr	Hurricane	729 (12:09)	★	−105 (1:45)	−132 (2:12)	Buddy 24
MD	Mechanical Drone	834 (13:54)	+105 (1:45)	★	−27	See in Sea 30
Shin	Shindig	861 (14:21)	+132 (2:12)	+27	★	Raider 28

Hurr		MD	Shin
1/3	4:03	+35	+44
2/3	8:06	+1:10	+1:28
1	12:09	+1:45	+2:12
2	24:18	+3:30	+4:24
3	36:27	+5:15	+6:36
4	48:36	+7:00	+8:48
5	1:00:45	+8:45	+11:00
6	1:12:54	+10:30	+13:12
7	1:25:03	+12:15	+15:24
8	1:37:12	+14:00	+17:36
9	1:49:21	+15:45	+19:48

MD		Shin	Hurr
1/3	4:38	+9	−35
2/3	9:16	+18	−1:10
1	13:54	+27	−1:45
2	27:48	+54	−3:30
3	41:42	+1:21	−5:15
4	55:36	+1:48	−7:00
5	1:09:30	+2:15	−8:45
6	1:23:24	+2:42	−10:30
7	1:37:18	+3:09	−12:15
8	1:51:12	+3:36	−14:00
9	2:05:06	+4:03	−15:45

Shindig		MD	Hurr
1/3	4:47	−9	−44
2/3	9:34	−18	−1:28
1	14:21	−27	−2:12
2	28:42	−54	−4:24
3	43:03	−1:21	−6:36
4	57:24	−1:48	−8:48
5	1:11:45	−2:15	−11:00
6	1:26:06	−2:42	−13:12
7	1:40:27	−3:09	−15:24
8	1:54:48	−3:36	−17:36
9	2:09:09	−4:03	−19:48

Part II

Two-Factor Handicapping Executive Summary

Chapter 3

Handicapping Today and the Wider Context

3.1 The Need for Equitable Handicapping

One-design racing aspires to the ideal of completely fair racing where every boat has the same potential to win in any given race. And it is accepted that, when boats of different design race against each other using a handicapping formula, this ideal can never be reached. The PHRF and IRC systems are pessimists, they embrace this *dismal truth*, employ this simplest possible handicapping formulas and encourage long racing series on a standardized course so things average out in the end. The IMS system was the first handicapping optimist; it used a *velocity prediction programme* (VPP) to predict a boat's performance in all conditions, on every point of sail and in every wind strength. This prediction profile could be exquisitely precise and the resulting handicapping "formula" was not a simple arithmetic operation but a complex interpolation scheme relying on a computer to determine corrected times.

IMS could be used on a single race with greater confidence in the fairness of the handicapping than PHRF could enjoy over a series of races by virtue of the greater precision of the prediction profile. A PHRF handicap is a single number which is necessarily an aggregate of performance on many point of sail across many different winds. In any given race between a pair of boats using a single-factor handicap the conditions will almost always favour one boat over the other. The limited precision in using a single number as a handicap guarantees that. Consider a statement like so "A C&C 30 is a good boat in heavy air and the Capri 25 a good boat in light air." Now they both have the same PHRF rating and on average they have similar performance. The one handicapping factor represents the average performance of the boats — it is a first-order term in the mathematical model of the boats' performance. In a single-factor handicap there is simply no place to represent the extra information of how the boats perform differently away from the average. A second-order term could quantify just how good a C&C 30 is in heavy air — handicaps with at least two factors are the necessary to model such a simple statement.

The ORC and ORR systems are the surviving handicapping optimists. IRC was the diehard pessimist's answer to IMS. These and every other modern system embrace the extremes: the handicapping formula is either very simple and very imprecise or it is very complex and very precise. The middle ground between simplicity and precision has been vacated. Two-factor time-on-time-and-distance handicapping sits in this Goldilocks zone of not too imprecise and yet not too complex. But no system is currently employing time-on-time-and-distance handicapping.

3.2 A Summary of the Argument for Time-on-Time-and-Distance

ORC and ORR temper their handicapping optimism by offering single-factor handicaps for simple races, either on a club certificate or as an alternative on the full certificate. What a waste. Time-on-time-and-distance handicapping, with its two handicapping factors, offers clear advantages for race organizers and competitors alike, and should be the natural fallback position from a full performance curve. ORR (in its guise as Americap) and ORC have trod this ground before but, through various missteps, have always found the middle ground between imprecision and complexity rather than a robust trade-off between simplicity and precision.

- Recognizing where these trade-offs occur is the key to coming up with a club racing handicap which is both robust in the face of changing circumstances yet simple enough for race organizers to use.
- For competitors in an event using the full power of ORC and ORR, performance curves must be issued for each race — in the vast majority of cases issuing time-on-time-and-distance handicaps could do the job just as well while putting much less demand on the competitors themselves.
- In no way should the club race and the grand-prix race be using the same time-on-time-and-distance handicap. The demands of club-racing robustness and grand-prix accuracy are at odds with each other — that the two-factor time-on-time-and-distance can embrace both positions is a testament to the strength of the handicapping method — but conflating the two concerns by using the exact same handicap cannot succeed.
- Handicapping using time-on-time-and-distance can completely subsume either time-on-distance or time-on-time as a special case. Likewise using a performance curve can completely subsume time-on-time-and-distance as a special case. But time-on-distance, time-on-time and time-on-time-and-distance can all be reckoned using simple proportions. Trying to compare general performance curves using only mental arithmetic would be fruitless.
- A single-factor time-on-distance or time-on-time system lacks the precision to handicap any one race fairly. Even given the constraints on race organization at the club level, an appropriately designed time-on-time-and-distance handicapping system cannot fail to be fairer than either. All this without the added organizational burden of determining the appropriate wind range prior to racing.

A rational observer would conclude that a time-on-time-and-distance calculation of corrected times should be the norm which is deviated from only in unusual circumstances. There may be venues where the wind is so predictable that the simplicity of time-on-distance or time-on-time handicaps outweigh their limitations. Or there may be events where the full precision of performance curves is worth the added complexity. Either extreme is not worthy of determining everyday usage.

3.3 The Inputs to the Corrected Time Formula

In discussing how a race is handicapped it is necessary to distinguish between the subjective *specialization* of a handicap by race officials before a race begins and the objective application of the resulting handicap in a corrected time formula once the race is under way. Note that IRC and PHRF offer no specialization to race organizers. The single handicap on the rating certificate is what you get, regardless of course configuration or wind speed. The Portsmouth Yardstick has variant time-on-time handicaps for different wind ranges based on the Beaufort scale. In order for competitors to use the correct handicap on the water the Beaufort number needs to be announced before racing begins.

ORC has a club-racing certificate that requires choosing between time-on-distance and time-on-time variants and between long-distance and windward-leeward variants. Having selected one of these four choices the next decision is between a single-number versus a triple-number handicap. All these choices would be published in the notice of race. The triple-number handicap has three variants for light, medium and heavy air and must be specialized down to a single number by the subjective observation of a race official just before the race starts. The result is a single-factor handicap which can be applied to a corrected time formula by racers throughout and by race officials at the finish line.

Performance curve scoring under ORC or ORR integrates, using the distance and expected wind direction on each leg of the course, the expected elapsed times at each of several predetermined wind speeds. The resulting *IMS-style performance curve* is a function which interpolates between these predetermined wind speeds to continuously map wind speed to elapsed time. The whole performance curve then serves as the handicap for the corrected time formula. Note that the table of paces at different wind angles and wind speeds on the handicapping certificate is specialized to a single number at each wind speed before racing. During racing an appropriate wind speed is inferred inside the corrected time formula.

In order to be truly effective for racers and race officials alike, a boat's corrected time can only depend on its specialized handicap, a scratch handicap, the course distance and its own elapsed time. The specialized handicap may contain subjective elements but these are known to all competitors before racing. Course distances are the same for everyone. The scratch handicap is the same for everyone and is nominally part of every corrected time formula but this is mostly a fudge factor to make corrected times look pretty — any choice of scratch handicap will always yield the same order of finishes. And a boat's own elapsed time is the only observation that is objective and always available to it. Any handicapping formulation that meets these criteria can be used by boats to judge their own performance against nearby competitors throughout a race. Handicapping systems have failed in this regard — and not just in the past. ORC made a change to its performance curve scoring formula in 2017 which seriously compromises its use on the race course, making everyone's corrected time dependent on the elapsed time of the winning boat! This was a serious mistake suddenly introduced after many years of doing the right thing.

3.4 The General Corrected Time Formula

While it may not be immediately obvious, corrected time formulas which meet the criteria stated above are all ultimately the same. Or rather, they are instances of a general and highly abstract formulation which encompasses them all. Any handicap can be presented in such a way that it applies to the general formula. The most common and concrete incarnations are:

Level Racing — the Zero-Factor Handicap is one that has no effect.

Single-Factor Handicapping — Time-on-Distance and Time-on-Time

> Both time-on-distance and time-on-time require a single number as a handicap. For time-on-distance the time allowances are proportional to course distance. For time-on-time the time allowances are proportional to elapsed time.

The Two-Factor Synthesis — Time-on-Time-and-Distance

> Time-on-time-and-distance requires two numbers as a handicap. One of the factors is a clear analogue to a time-on-distance handicap and the other to a time-on-time handicap. Put together they are quite a bit more powerful than either alone yielding a handicap which, like a performance curve, is inherently responsive to changes in wind speed. And, like its antecedents, time allowances can be built-up proportionally making them amenable to mental arithmetic.

A Multi-Factor Abstraction — the IMS Style Performance Curve

In the abstract a performance curve is a continuous function of wind speed but in practice it is actually parametrized by just a few numbers. More factors refine the curve to better model the boats' actual performance — each additional factor provides more precision but less improvement in accuracy.

Unlike the previous schemes, performance curves are not at all amenable to mental arithmetic. It is still possible for competitors to estimate time allowances on the water, but precomputed tables and a good head for interpolation are needed.

The Ultimate Abstraction — the General Performance Curve

Having wind speed as the domain of performance curves is a hindrance to an efficient parametrization. The most general abstraction abandons the explicit dependence on wind speed. Handicaps are then considered a relationship between a boat's pace and a generalized variable that can be compared to wind speed in an order preserving way.

The more general time-on-time-and-distance handicapping can be used to model either time-on-distance or time-on-time as a special case. Likewise the general performance curve can encompass either time-on-time-and-distance handicapping or IMS-style performance curves. Intermediate in complexity between time-on-time-and-distance and the general model are polynomial or spline models based on three or four parameters — this is fewer than the seven parameters typical for an IMS-style performance curve but should still be more accurate. Each of these layers of generality encompasses the earlier and improves precision and accuracy — but does so at ever decreasing margins of improvement. There is clearly a sweet spot — a point at which having fewer handicapping factors is too imprecise and yet having more handicapping factors is too complex.

Note that this sort of abstract generality is very nice for an analysis of handicapping — the regression model for computing multi-factor performance handicaps pops out quite nicely — but it is very removed from practice and can distract from the inherent simplicity of time-on-distance, time-on-time and time-on-time-and-distance handicapping.

3.5 Time-on-Distance versus Time-on-Time

Time-on-distance and time-on-time are both, each in their own way, reasonable first-order models of how boats perform on the race course. Because of having only one handicapping factor they cannot explicitly model how a boat's performance changes in different winds; instead, they implicitly model how *every* boat's pace *must* change relative to competing boats. In the time-on-distance model time allowances remain constant throughout a race whereas in the time-on-time model they increase proportionately as the race progresses in time. For any given pair of boats one or the other model will better approximate their relative performance through varying winds but, for an entire fleet of boats scaled to different lengths but of a similar design, it is known that time-on-time does a better job.

But the choice between time-on-distance and time-on-time has rarely been based on accuracy or predictive power. Time-on-time has long dominated in dinghy racing around dropped marks where course lengths were often not measured. Time-on-distance has been very popular in North America for offshore boats sailing on fixed courses of fixed length. Under the old IOR rule rated lengths were converted to time-on-distance handicaps in North America and time-on-time handicaps in the rest of the world. Local conventions are being slowly overturned with a move toward global rating systems but, with ORC Club having embraced time-on-whatever agnosticism, *slowly* is the operative word.

Contrarily, the US based ORR now recommends time-on-time handicapping for events which do not take advantage of performance curve scoring.

The perceived need for triple-number handicaps in events which do not warrant the complexity of performance curve scoring underline the common weakness of single-factor handicaps, the predictable outcome of races depending on the wind. It should be noted that if the wind has been accurately predicted then so have the expected elapsed times for boats; as a result time allowances calculated using wind-specialized time-on-time handicaps will closely correspond to the time-allowances calculated using wind-specialized time-on-distance handicaps. The choice between time-on-distance and time-on-time becomes moot once you have chosen to use a wind-specialized handicap, so long as the wind co-operates — each handicap variant is independent of the others and cannot influence the outcome of the race once the decision to specialize on the wind range has been made. And it becomes troublesome at the boundary between light-to-medium or medium-to-heavy air. How and when is this determination made? Leaving the decision until after boats have finished would leave competitors in a weird quantum superposition of races, like Schrödinger's cat, uncertain of the outcome until the lid on the box is raised. Multi-factor handicaps don't suffer from these uncertainties.

3.6 The Time-on-Time-and-Distance Synthesis

Two-factor time-on-time-and-distance handicapping combines elapsed time and the course distance very simply in the corrected time formula to respond linearly to varying winds in a single all-encompassing regime. Time allowances are the sum of a constant base together with an offset proportional to the time behind or ahead of schedule. Although it can be thought of as a synthesis of time-and-distance and time-on-time, and algebraically it very much is, in practice such handicapping acts like a first-order model together with a second-order correction to respond to the changes in the wind.

3.6.1 Common Weaknesses of All Multi-Factor Corrected Time Formulas

All multi-factor corrected time formulas infer wind speed from a boat's own elapsed time and, as such, are subject to all the possible errors that can confabulate this inference: using a handicap inappropriate for the course configuration, currents, wind shifts that turn beats into reaches and vice versa, pockets of dead air, traffic jams at mark roundings, et cetera. And although these erroneous influences also effect time-on-distance and time-on-time handicapping, the second-order corrections implicit in any multi-factor handicap will compound these errors and can, if the second-order corrections are sufficient to meet the theoretically perfect need, greatly exaggerate them.

Some of the largest sources of possible error in inferring wind speed come from the use of an inappropriate handicap for the points of sail on the course and from failing to take currents into account. Performance curves are always specialized in this manner but for seemingly simpler time-on-time-and-distance handicapping these steps might be skipped to deleterious effect. But at a particular venue or for a particular style of course, for windward/leeward courses in a low current area say, a fixed time-on-time-and-distance handicap might be used with confidence. For a predictably repeatable course configuration with reliable winds and current, a single performance curve or time-on-time-and-distance handicap is all that is needed.

Given that the errors associated with applying second-order corrections are so much greater than in applying the first-order modelling it might make sense to reduce second-order effects. This applies equally to performance curves as it does to time-on-time-and-distance handicaps but, given the friendlier nature of time-on-time-and-distance handicapping, it seems more relevant in the latter case.

This would result in handicapping closer to time-on-time in its robustness while still including some sensitivity to the wind — in terms of precision, wind-deadened time-on-time-and-distance can be thought of as factor-and-a-half handicapping. Trading accuracy for robustness could be desirable in many contexts. In club racing it might be appropriate to reuse an wind-deadened unspecialized handicap in lieu of specializing a wind-sensitive handicap to account for erroneous influences. Using a light-medium-heavy air triple of wind-deadened handicaps sounds crazy but it maintains high accuracy under a broad range of conditions while having little up front complexity for race officials.

3.6.2 Planing Boats

A two-factor handicap has a single degree of freedom to represent the second-order correction that responds to the changes in the wind. For planing boats this simply isn't enough to represent the transition from displacement to planing mode. A performance curve will do better but that too is usually parametrized for a smooth curve, whereas the planing boat will have a sharp kink in its performance curve. Very few handicapping systems even attempt to accurately model the planing boat. Note that none of the offshore system in current use attempt to tackle this problem. The time-on-time-and-distance corrected time formula can not.

3.6.3 The Power of Two-Factor Handicaps

Despite the potential pitfalls, time-on-time-and-distance still offers enormous advantages over either time-on-distance or time-on-time. Multi-factor handicaps give rating authorities and race officials the power to shoot themselves in the foot — a single factor handicap only arms a race organizer with nerf balls, safe but broad and fuzzy — but years of accumulated best practices with performance curve scoring have made clear how to use them well. And the advantages to competitors in being able to use a simple time-on-time-and-distance handicap make it unclear when the marginal improvement in accuracy given by a performance curve make up for the inconvenience. Two factors occupies a sweet spot for club-level and national events alike.

VPP based rating rules would do well to always specialize down to a time-on-time-and-distance handicap. For national level races in venues with significant currents or with peculiar course configurations a handicap should be specialized from a complete profile in the same manner as a performance curve. Precomputed variants for windward/leeward, triangular or circular courses could serve in all other situations.

Performance based handicapping systems rarely collect data sufficient to be able to compile a complete performance profile. While theoretically possible it would require either detailed per leg information or a very large sample over a limited number of well identified course configurations to prize out the polar diagrammes. In practice it would make more sense to combine a generic set of polars with race results to deduce a boat's performance profile. And it would be easy to compute a wind-deadened handicap for club series races on a generic closed course.

At a particular venue it could be advantageous to compute per-boat performance handicaps for those that regularly race. The data will be more consistent and probably more reliable than that available to a local handicapping authority.

3.6.4 Americap

One can't talk about time-on-time-and-distance handicapping without bringing up how badly Americap failed in the US. Americap was a VPP based two-factor handicap and theoretically very well founded. Had the VPP based performance profile been tweaked to make it easy to specialize into an event specific time-on-time-and-distance handicap and this used instead of a seven-factor performance curve it would have been a great success. Performance curves are hard for competitors to use on the water and the ease in dealing with the greatly simplified time allowances would have more than made up for any marginal losses in precision and accuracy, even in the highly competitive world where these kind of yachts race. And no doubt interest about two-factor handicaps would have trickled down to club level events. But Americap was pushed as a kind of super PHRF, a be-all-and-end-all two-factor handicap that would supersede the need for specialized handicapping. No handicap can meet those goals and, even then, it was barely given a chance to fail.

The other problem with Americap was more technical. The *time correction factor* (TCF) is a blight that now infects time-on-time handicapping. This is one of those disasters that at first sight seems sensible — take the reciprocal of a Portsmouth Yardstick style DN to turn a divisor into a multiplier and corrected times become easier to calculate and every one wins — until you try to work out time allowances on the water and suddenly you need a hand calculator. Now the Portsmouth Yardstick had already got this right; a Portsmouth DN can be used in a very simple proportionality to work out time allowances on the fly. The Americap corrected time formula was a true synthesis of a time-on-time TCF-style formula and a standard time-on-distance formula — it literally just added the two terms together. And, like a TCF but more so, the resulting two-factor handicaps are absurdly inconvenient to use on the water. And this *is* absurd as the foremost benefit of a time-on-time-and-distance handicap over a performance curve is the ease with which boats can quickly judge how they are performing relative to their competitors. Having got the highly technical math and physics right in the VPP simulation, the designers of Americap fumbled at the grade-school level arithmetic necessary to turn their handicaps into a viable system on the water.

3.7 Performance Handicaps

3.7.1 Upgrading an Existing Handicapping Scheme

All existing performance handicaps are either time-on-distance or time-on-time in their derivation (and only reluctantly support the alternative style of handicapping). Without supplementary data it is meaningless to upscale a PHRF handicap into a time-on-time-and-distance handicap. That is, it easy to embed an existing time-on-distance or time-on-time system into a time-on-time-and-distance one, but the resulting handicapping will behave just like the underlying system. With only the first-order data that an existing handicap system provides there is no data to supply the second-order corrections that a two-factor handicap needs to be effective. Now the Portsmouth Yardstick does provide a source of second-order corrections with their Beaufort-number specific handicapping but, given how the Yardstick is favoured for use with planing dinghies and the greatest weakness of using only a single second-order factor is how hopeless it is for modelling the shift from displacement to planing mode, this data is not as useful as it might at first seem.

On the other hand, given modern tools, it is easy to compute multi-factor handicaps directly from race results in toto. Number crunching a matrix with thousands of rows for boat classes and tens of thousands of columns for races is considered a small problem. All the difficulty comes from accumulating the data. A reliable database of race data would be a great boon to anyone interested in performance handicapping. Nonlinear regression modelling for handicapping schemes of arbitrary complexity can be accomplished using only free software on a cast-off personal computer.

3.7.2 The Statistical Model and the Optimization Problem for Computing Handicaps

- there is a straightforward statistical model for pace with a normal error term
- for large and medium size data sets the normal error model works extremely well
- it yields a sum-of-squares optimization problem between observed pace and predicted pace
- it has two free variables for each handicap class and one free variable for each race
- and is under-determined unless a handicapping gauge is also specified

This is a non-linear optimization problem and requires an iterative solver, typically Gauss's method. But a ping-ponging back and forth between the free variables associated with each handicap class and the free variables associated with each race is so simple and fast that it may be unwarranted to use a more sophisticated solver.

3.7.3 Statistical Inference on Performance Data

The time-on-time-and-distance model has submodels for time-on-distance and time-on-time. Time-on-distance has a linear model equivalent to a two-way analysis of variance problem. The other two models are non-linear, but infer the same statistics as you would expect to see with a linear model. We can calculate R^2 and F statistics between the time-on-time-and-distance model and its submodels to infer its efficacy over each.

Chapter 4

Time-on-Time-and-Distance Exposition

4.1 The Two Factors of the Time-on-Time-and-Distance Handicap

A two-factor handicap $\begin{bmatrix}k & h\end{bmatrix}$ consists of a time-on-time factor k and a time-on-distance factor h both of which are measured in seconds per nautical mile. The h factor is the boat's pace on average. Note that a slower pace is represented by a greater number of seconds per mile. The k quantifies how the seconds per mile should increase as the wind decreases; when we expect a race to run 10% slower than average we would expect the pace for a particular boat to be $h + 10\%k$ based upon its two-factor handicap. We will write this relationship algebraically as

$$\hat{p} = h + k{\cdot}q$$

where q is the hypothetical percentage of time slower than average for the race and $\hat{p}$ (with a *hat* on top and read "p-hat") is the expected pace in seconds per nautical mile — note that a negative q corresponds to a faster-than-average race.

We can compare each handicapped boat's expected pace at a given value of q. Fixing $q = 0$ we fall back to a single-factor handicapping scheme with h the only factor; in this context we will denote the handicap g and call it a general purpose handicap, a handicap which can be used for either time-on-distance or time-on-time handicapping. For a given value of q the expected pace for each boat $\hat{p}$ can be thought of as a general purpose handicap as specialized to the wind. There is a simple conversion $\begin{bmatrix}k & h\end{bmatrix} \leftrightarrow \begin{bmatrix}g_{\text{light}} & g_{\text{heavy}}\end{bmatrix}$; we fix the value of $q = \pm{}^{1}\!/_{3}$ to break the expected pace into ranges

$$\begin{aligned} g_{\text{light}} &= h + {}^{k}\!/_{3} \\ g_{\text{heavy}} &= h - {}^{k}\!/_{3} \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} k &= \tfrac{3}{2}(g_{\text{light}} - g_{\text{heavy}}) \\ h &= \tfrac{1}{2}(g_{\text{light}} + g_{\text{heavy}}) \end{aligned}$$

4.2 Reckoning Time Allowances from Our Point of View

In order to compare ourselves against a competitor on-the-water, we need to reckon a pace allowance Δp (read "delta-p") or a time allowance Δt (read "delta-t") between *their boat* and *ours*. Our boat has the handicap $\begin{bmatrix}k & h\end{bmatrix}$ and their boat has a handicap which differs from ours by $\begin{bmatrix}\Delta k & \Delta h\end{bmatrix}$ (read as "the pair delta-k and delta-h"). For our boat the predicted pace $\hat{p}$ (with a *hat* on top — read "p-hat") in excess of (i.e. numerically larger than) h (actually slower) can be written

$$\hat{p} - h = k{\cdot}q$$

We use the variable $\Delta\hat{p}$ (read "delta-p-hat") to denote the difference in predicted pace between their boat and ours. The part in excess of Δh can be written

$$\Delta\hat{p} - \Delta h = \Delta k{\cdot}q$$

For those a little rusty in algebraic convention $\Delta k{\cdot}q$ is "delta-k" times "q" — "delta-k" is just a two letter long variable name where the "delta" in the name is suggestive of a difference — the dot unambiguously delineates the two variables names which are then implicitly multiplied together. We see that the relationship between our actual pace p in excess of h and the predicted allowance for their pace Δp in excess of Δh can be expressed as a proportionality by elimination on the free variable q

$$\begin{aligned}
\Delta p - \Delta h \;:\; \Delta k &\overset{\text{in proportion}}{=} p - h \;:\; k && \text{(a pace allowance)}\\
\Delta t - \Delta h{\cdot}d \;:\; \Delta k{\times}1\,\text{mile} &\overset{\text{in proportion}}{=} t - h{\cdot}d \;:\; k{\times}1\,\text{mile} && \text{(a time allowance, or...)}\\
\Delta t - \Delta h{\cdot}d \;:\; \Delta k{\times}{}^{1}\!/_{3}\,\text{mile} &\overset{\text{in proportion}}{=} t - h{\cdot}d \;:\; k{\times}{}^{1}\!/_{3}\,\text{mile} && \text{(with finer gradations)}
\end{aligned}$$

The time allowance is comprised of two parts: the fixed part $\Delta h{\cdot}d$ at time $h{\cdot}d$ and the excess part Δk for each interval of time k which departs from $h{\cdot}d$. For comparison purposes, let our boat have the single-factor general-purpose handicap g and their boat have a corresponding handicap which differs from ours by Δg. The time-on-time proportionality is a little easier to deal with as there is no fixed part

$$\begin{aligned}
\Delta p \;:\; \Delta g &\overset{\text{in proportion}}{=} p \;:\; g && \text{(a pace allowance)}\\
\Delta t \;:\; \Delta g{\times}1\,\text{mile} &\overset{\text{in proportion}}{=} t \;:\; g{\times}1\,\text{mile} && \text{(a time allowance, or...)}\\
\Delta t \;:\; \Delta g{\times}{}^{1}\!/_{3}\,\text{mile} &\overset{\text{in proportion}}{=} t \;:\; g{\times}{}^{1}\!/_{3}\,\text{mile} && \text{(with finer gradations)}
\end{aligned}$$

and the time-on-distance handicap is the simplest of all as there is only a fixed part

$$\begin{aligned}
\Delta p &= \Delta g && \text{(a pace allowance)}\\
\Delta t &= \Delta g{\cdot}d && \text{(a time allowance)}
\end{aligned}$$

4.3 Applying the Time-on-Time-and-Distance Handicap to Rank Finishers

4.3.1 Placing Boats via the Performance Prediction Relationship of a Handicap

It is not immediately obvious how a race committee should apply a handicap $\begin{bmatrix}k & h\end{bmatrix}$ to a given race when its predictive value relies on an unknown variable q but, having investigated time allowances, it should be clear how to proceed. We will start with a boat's observed course-average pace p which is simply its elapsed time t in seconds divided by the course distance d in nautical miles. We then work backwards from p to calculate $\check{q}$ (with a *check* on top — read "q-check") its percentage slower or faster than average with regard to its own handicap $\begin{bmatrix}k & h\end{bmatrix}$. This is the value the variable q would take in order for the predicted pace $\hat{p}$ to match the observed pace p. Algebraically this is called a preïmage

$$p = h + k{\cdot}\check{q} \quad\iff\quad \check{q} = \frac{p-h}{k}$$

The mapping $p \mapsto \check{q}$ is the functional inverse to the mapping $q \mapsto \hat{p}$. The preïmage $\check{q}$ can be compared between boats just like a corrected time to determine the handicapped finish order of boats. Even though the definition of the time-on-time-and-distance handicap relies on a free variable q to come to a performance prediction there are no external variables in the application of the handicap to a particular race.

4.3.2 And on Consistency with Time Allowances

Boats that finish with the same $\check{q}$ will have finish times that differ by their time allowances — in either case the free variable q is eliminated in an algebraically equivalent way — so our formulation to rank boats is consistent with our predictions and the time allowances derived from them.

4.3.3 Corrected Times, the Scratch Boat ★ and its Handicap $\begin{bmatrix} {}^{\bigstar}k & {}^{\bigstar}h \end{bmatrix}$

We can already order finishes using the $\check{q}$ so further calculations seem redundant; however the Racing Rules of Sailing require a corrected time for each finisher. To do this we need to select a representative type of boat which we call the scratch boat together with its handicap $\begin{bmatrix} {}^{\bigstar}k & {}^{\bigstar}h \end{bmatrix}$. We will then calculate the corresponding pace for each boat were it also of the scratch type. We will call this the corrected pace $\check{\check{p}}$ (with two checks on top). For each boat

$$\check{\check{p}} = {}^{\bigstar}h + {}^{\bigstar}k{\cdot}\check{q} \qquad\qquad \left(\text{so that} \qquad \check{q} = \frac{\check{\check{p}} - {}^{\bigstar}h}{{}^{\bigstar}k}\right)$$

To turn this into a corrected time $\check{\check{t}}$ we simply multiply out by course distance d

$$\check{\check{t}} = \check{\check{p}}{\cdot}d$$

The corrected times $\check{\check{t}}$ for different boats order exactly the same as the $\check{q}$ they are derived from.

4.3.4 And Again on Consistency

This prediction of how a boat should have finished were it of the scratch type, sorts boats identically to the underlying comparison of the $\check{q}$ and identically to the pairwise comparison of boats according to their mutual time allowances.

4.3.5 For Equivalent Formulations of Corrected Time

If we ignore the standard algebraic order of operations and simply write the arithmetic operations in a left-to-right fashion as sequential computations (instead of writing brackets)

$$\check{\check{p}} \;=\; \overbrace{p \;\; \underline{-\,h} \;\; \underline{\div\,k}}^{\check{q}} \;\; \underline{\times\,{}^{\bigstar}k} \;\; \underline{+\,{}^{\bigstar}h}$$

We just subtract h, divide by k, multiply by ${}^{\bigstar}k$ and add ${}^{\bigstar}h$ in sequence. In terms of elapsed and corrected time we first divide elapsed time by course length d, apply the calculations above then finally multiply again by d for the corrected time, so avoiding pace as anything but an intermediate state in the computation

$$\check{\check{t}} \;=\; \overbrace{t \;\; \underline{\div\,d} \;\; \underline{-\,h} \;\; \underline{\div\,k}}^{\check{q}} \;\; \underline{\times\,{}^{\bigstar}k} \;\; \underline{+\,{}^{\bigstar}h} \;\; \underline{\times\,d}$$

The computed corrected time $\check{\check{t}}$, elapsed time t and handicap $\begin{bmatrix} k & h \end{bmatrix}$ vary from boat to boat. The scratch handicap $\begin{bmatrix} {}^{\bigstar}k & {}^{\bigstar}h \end{bmatrix}$ and course length d (used twice per computation) are common to all the boats.

4.3.6 A Further Simplification of Corrected Time for Fleet Computations

To simplify the above formulation this further we fall back to normal algebraic notation and write defining equations with a common term ${}^\bigstar h{\cdot}d$

$$\check{\check{t}} \;=\; {}^\bigstar h{\cdot}d + {}^\bigstar k\,\frac{t - h{\cdot}d}{k}$$

The ${}^\bigstar h{\cdot}d$ term can be precomputed thereby dropping the number of arithmetic operation per boat from six to five — but this is a petty optimization; rather, it's nice to be able to define corrected time without regard to a corrected pace; although pace allowances and corrected paces arise very naturally in the development and the graphical interpretation of handicapping, they aren't very useful in their application to an individual race. Elapsed times and time allowances better indicate how competitively a boat is performing within a given race. In our worked examples we will compute paces only for expository purposes.

4.3.7 A Nice Symmetric Defining Equation for Corrected Time

The following defining equation beautifully describes the relationship between elapsed and corrected time by sacrificing the explicit formulation of the solution

$$\frac{\check{\check{t}} - {}^\bigstar h{\cdot}d}{{}^\bigstar k} \;=\; \frac{t - h{\cdot}d}{k}$$

As a defining statement of the handicapping relationship, this equation must be solved for the unknown $\check{\check{t}}$ given all the other known values. Note the similarity to the proportionality that was used to determine a time allowance. As in that treatment, we can imagine that an unknown is eliminated by equating the left and right sides; the corrected form on the left and the observed form on the right.

Because this definition is an equation of ratios, it would be tempting to call this a proportionality in its own right; however the units are not the same in the numerator as in the denominator. Were we to balance those units then the eliminated unknown (in the middle) would be q (solving to preïmage $\check{q}$)

$$\frac{\check{\check{t}} - {}^\bigstar h{\cdot}d}{{}^\bigstar k{\cdot}d} \;=\; \check{q} \;=\; \frac{t - h{\cdot}d}{k{\cdot}d} \qquad\qquad \left(\text{c.f.} \qquad \frac{\check{\check{p}} - {}^\bigstar h}{{}^\bigstar k} \;=\; \check{q} \;=\; \frac{p - h}{k}\right)$$

These handicapping factors multiplied by course distance (called *course-specific handicapping factors*) will be in units of time and, although unneeded for scoring or determining time allowances, can be useful for comparisons between different styles of handicapping.

4.3.8 Yet Another Equivalent Algebraic Determination of Corrected Time

Given the known values of d, $\begin{bmatrix}{}^\bigstar k & {}^\bigstar h\end{bmatrix}$, $\begin{bmatrix}k & h\end{bmatrix}$ and t we can solve for the two unknowns $\check{\check{t}}$ and u in this pair of equations

$$\begin{aligned} \check{\check{t}} &= {}^\bigstar k{\cdot}u + {}^\bigstar h{\cdot}d \\ t &= \;\; k{\cdot}u + \;\; h{\cdot}d \end{aligned}$$

We begin by solving for u in the second equation and then use that to solve for $\check{\check{t}}$ in the first equation. Solution u is discarded while solution $\check{\check{t}}$ is the desired corrected time. This differs from the explicit formulation only in having a prettier presentation — solving this system of equations just recapitulates what we have already seen.

4.4 Example Boats and their Handicaps

Let's see some examples with *Mechanical Drone* as our boat and with *Hurricane*, *Winged Elephant*, *Shindig*, *Professor* and *Rhumb Punch* as our competitors.

An Example Handicap for Our Boat *Mechanical Drone* $\begin{bmatrix} k & h \end{bmatrix} = \begin{bmatrix} 813\,\text{s/mile} & 840\,\text{s/mile} \end{bmatrix}$

The q variable can range from about -33% to $+100\%$ or more depending on how slow a race committee allows a race to get before abandoning it. For our example boat $q = +100\%$ would yield a pace of 1653 s/mile which corresponds to a 2.2 kt speed. By thirds

$$
\begin{array}{llll}
q = -1/3 \implies \hat{p} = h - k/3 & = 840\,\text{s/mile} - 271\,\text{s/mile} & = 569\,\text{s/mile}\,(6.3\,\text{kt}) & \text{heavy air pace} \\
q = 0 \implies \hat{p} = h & = 840\,\text{s/mile} & = 840\,\text{s/mile}\,(4.3\,\text{kt}) & \text{middling pace} \\
q = +1/3 \implies \hat{p} = h + k/3 & = 840\,\text{s/mile} + 271\,\text{s/mile} & = 1111\,\text{s/mile}\,(3.2\,\text{kt}) & \text{light air pace} \\
q = +2/3 \implies \hat{p} = h + 2k/3 & = 840\,\text{s/mile} + 542\,\text{s/mile} & = 1382\,\text{s/mile}\,(2.6\,\text{kt}) & \text{very light air pace} \\
q = +3/3 \implies \hat{p} = h + k & = 840\,\text{s/mile} + 813\,\text{s/mile} & = 1653\,\text{s/mile}\,(2.2\,\text{kt}) & \text{abandon-this-race pace}
\end{array}
$$

It is often more convenient, especially on the water when dealing with time allowances, to state handicaps in units of minutes and seconds per mile rather than as plain seconds per mile. In this reckoning $\begin{bmatrix} 813\,\text{s/mile} & 840\,\text{s/mile} \end{bmatrix} = \begin{bmatrix} 13\,\text{min}\,33\,\text{s/mile} & 14\,\text{min/mile} \end{bmatrix}$. When units can be omitted (e.g. in a table) we would express this more succinctly as [13:33 14:00].

4.4.1 More Example Handicaps

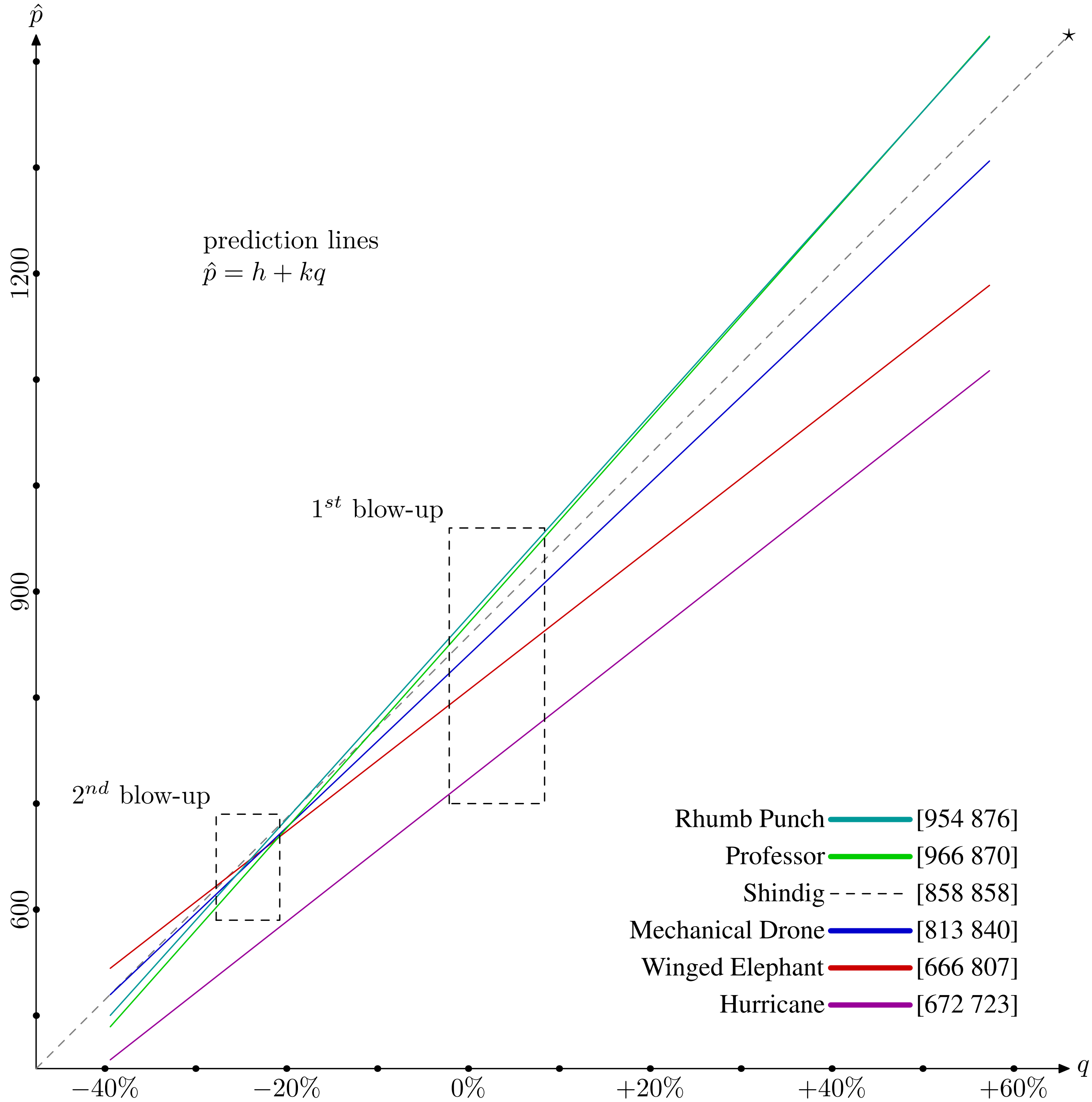
Here are some more two-factor $\begin{bmatrix} k & h \end{bmatrix}$ performance handicaps. Entries in the principal units of seconds per mile (s/mile) are duplicated as minutes and seconds per mile (min : s/mile) in the adjacent parenthesized expressions. The "deltas" Δk and Δh are the differences from the k, h of our boat *Mechanical Drone* (as indicated by a ∘).

Example Boat	[k h]	[Δk Δh]	Make
Hurricane	[672(11:12) 723(12:03)]	[−141(2:21) −117(1:57)]	Buddy 24
Winged Elephant	[666(11:06) 807(13:27)]	[−147(2:27) −33]	Frequency 24
Mechanical Drone	[813(13:33) 840(14:00)]	∘	See in Sea 30
Shindig	[858(14:18) 858(14:18)]	[+45 +18]	Raider 28
Professor	[966(16:06) 870(14:30)]	[+153(2:33) +30]	Stone 22
Rhumb Punch	[954(16:54) 876(14:36)]	[+141(2:21) +36]	Chimera 33

h by itself can also serve as a single-factor handicap for either time-on-distance or time-on-time (in this context the single-factor handicap will be denoted g for general-purpose). For a handicap derived from a VPP the distance coefficient of a time-on-time-and-distance handicap, the single factor of a time-on-time and the single factor of a time-on-distance handicap will all be identical. But for a performance handicap there can be small differences in these factors, especially for a small sample set. Unless each boat has performance data from races that match the overall distribution of wind conditions we should expect small discrepancies between the h factor of a two-factor and the g of a single-factor performance handicap.

4.4.2 Visualizing Handicaps with *Shindig* as Scratch Boat

We can visualize time-on-time-and-distance handicaps by graphing the prediction lines. We have $\hat{p}$ on the vertical axis and q on the horizontal axis. We lay the prediction line for the scratch boat *Shindig* (as indicated by a ★) along a 45° diagonal linking the scale on the two axes. All the plotted handicaps yield straight lines. Faster paces are down and to the left, slower paces up and to the right.



On a vertical line (at a particular value of q) the differences between the lines are the pace allowances between the boats as appropriate for the predicted $\hat{p}$. Note that a time allowance difference in finish time between boats requires that their corrected times be the same — using the prediction inherent in a handicap to derive a time allowance derives a corrected time formula to match.

The blow-ups indicated on the graph will be used in the two worked examples below: the 1st blow-up in 4.5.3 An Example Time-on-Time-and-Distance Race with *Shindig* as Scratch on page 44 and the 2nd blow-up in 4.5.4 Another Example with More Wind on page 45.

4.5 Worked Examples

Let's work some examples with the boats we introduced earlier. Our boat is ∘ *Mechanical Drone* for the purpose of reckoning time allowances and ★ *Shindig* is the scratch boat for purposes of graphing prediction lines and calculating corrected times.

4.5.1 Reckoning Time Allowances from Our Boat's Point of View

Our boat *Mechanical Drone* has handicap $\begin{bmatrix} k & h \end{bmatrix} = \begin{bmatrix} 813\,{}^{\text{s}}\!/\!_{\text{mile}} & 840\,{}^{\text{s}}\!/\!_{\text{mile}} \end{bmatrix} = \begin{bmatrix} {}^{13\,\text{min}\,33\,\text{s}}\!/\!_{\text{mile}} & {}^{14\,\text{min}}\!/\!_{\text{mile}} \end{bmatrix}$. Their boat *Shindig* has $\begin{bmatrix} \Delta k & \Delta h \end{bmatrix} = \begin{bmatrix} +45\,{}^{\text{s}}\!/\!_{\text{mile}} & +18\,{}^{\text{s}}\!/\!_{\text{mile}} \end{bmatrix}$. The pace allowance between theirs and ours will be expressed by a proportionality

$$\Delta p - {}^{18\,\text{s}}\!/\!_{\text{mile}} \;:\; {}^{45\,\text{s}}\!/\!_{\text{mile}} \quad \overset{\text{in proportion}}{=} \quad p - {}^{14\,\text{min}}\!/\!_{\text{mile}} \;:\; {}^{13\,\text{min}\,33\,\text{s}}\!/\!_{\text{mile}}$$

On a five mile course this would yield a proportionality for time allowances

$$\begin{aligned} \Delta t - {}^{18\,\text{s}}\!/\!_{\text{mile}} \times 5\,\text{mile} \;:\; {}^{45\,\text{s}}\!/\!_{\text{mile}} \times 1\,\text{mile} \quad &\overset{\text{in proportion}}{=} \quad t - {}^{14\,\text{min}}\!/\!_{\text{mile}} \times 5\,\text{mile} \;:\; {}^{13\,\text{min}\,33\,\text{s}}\!/\!_{\text{mile}} \times 1\,\text{mile} \\ \Delta t - 1\,\text{min}\,30\,\text{s} \;:\; 45\,\text{s} \quad &\overset{\text{in proportion}}{=} \quad t - 70\,\text{min} \;:\; 13\,\text{min}\,33\,\text{s} \end{aligned}$$

With a finer gradation (unit thirds in the second part of the ratio)

$$\begin{aligned} \Delta t - {}^{18\,\text{s}}\!/\!_{\text{mile}} \times 5\,\text{mile} : {}^{45\,\text{s}}\!/\!_{\text{mile}} \times {}^{1}\!/\!_{3}\,\text{mile} \quad &\overset{\text{in proportion}}{=} \quad t - {}^{14\,\text{min}}\!/\!_{\text{mile}} \times 5\,\text{mile} : {}^{13\,\text{min}\,33\,\text{s}}\!/\!_{\text{mile}} \times {}^{1}\!/\!_{3}\,\text{mile} \\ \Delta t - 1\,\text{min}\,30\,\text{s} : 15\,\text{s} \quad &\overset{\text{in proportion}}{=} \quad t - 70\,\text{min} : 4\,\text{min}\,31\,\text{s} \end{aligned}$$

The fixed part of the time allowance Δt that we give to their boat *Shindig* is $+1\,\text{min}\,30\,\text{s}$; this occurs at an elapsed time $t = 70\,\text{min}$ by our reckoning. Then for every $13\,\text{min}\,33\,\text{s}$ of t in excess of $70\,\text{min}$ the time allowance Δt increases by $+45\,\text{s}$. In unit thirds, for every $4\,\text{min}\,31\,\text{s}$ of elapsed time in excess of $70\,\text{min}$ the time allowance increases by $+15\,\text{s}$.

For *Professor* we have $\begin{bmatrix} \Delta k & \Delta h \end{bmatrix} = \begin{bmatrix} +153\,{}^{\text{s}}\!/\!_{\text{mile}} & +30\,{}^{\text{s}}\!/\!_{\text{mile}} \end{bmatrix}$; on the same five mile course

$$\begin{aligned} \Delta p - {}^{30\,\text{s}}\!/\!_{\text{mile}} \;:\; {}^{153\,\text{s}}\!/\!_{\text{mile}} \quad &\overset{\text{in proportion}}{=} \quad p - {}^{14\,\text{min}}\!/\!_{\text{mile}} \;:\; {}^{13\,\text{min}\,33\,\text{s}}\!/\!_{\text{mile}} \\ \Delta t - 2\,\text{min}\,30\,\text{s} \;:\; 153\,\text{s} \quad &\overset{\text{in proportion}}{=} \quad t - 70\,\text{min} \;:\; 13\,\text{min}\,33\,\text{s} \\ \Delta t - 2\,\text{min}\,30\,\text{s} \;:\; 51\,\text{s} \quad &\overset{\text{in proportion}}{=} \quad t - 70\,\text{min} \;:\; 4\,\text{min}\,31\,\text{s} \end{aligned}$$

The fixed part of the time allowance is $2\,\text{min}\,30\,\text{s}$. For every $4\,\text{min}\,31\,\text{s}$ in excess of $70\,\text{min}$ the time allowance increases by $+51\,\text{s}$.

We can repeat this calculation with the Δk and Δh for each of our competitors to determine all the time allowances we need; the ratio on the right-hand side of the proportionality is common to all boats.

4.5.2 Time Allowance Tables

To work with these allowances, we can use a pair of standard time allowance tables for the time and distance parts respectively. Our boat is MD†.

MD *on-time*		+Shin	+RP −Hurr	−WE	+Prof
1/3	04:31	15	47	49	51
2/3	09:02	30	1:34	1:38	1:42
1	13:33	45	2:21	2:27	2:33
1 1/3	18:04	1:00	3:08	3:16	3:24
1 2/3	22:35	1:15	3:55	4:05	4:15
2	27:06	1:30	4:42	4:54	5:06
2 1/3	31:37	1:45	5:29	5:43	5:57
2 2/3	36:08	2:00	6:16	6:32	6:48
3	40:39	2:15	7:03	7:21	7:39
3 1/3	45:10	2:30	7:50	8:10	8:30
3 2/3	49:41	2:45	8:37	8:59	9:21
4	54:12	3:00	9:24	9:48	10:12
4 1/3	58:43	3:15	10:11	10:37	11:03
4 2/3	1:03:14	3:30	10:58	11:26	11:54
5	1:07:45	3:45	11:45	12:15	12:45
5 1/3	1:12:16	4:00	12:32	13:04	13:36
5 2/3	1:16:47	4:15	13:19	13:53	14:27
6	1:21:18	4:30	14:06	14:42	15:18
6 1/3	1:25:49	4:45	14:53	15:31	16:09
6 2/3	1:30:20	5:00	15:40	16:20	17:00
7	1:34:51	5:15	16:27	17:09	17:51
7 1/3	1:39:22	5:30	17:14	17:58	18:42
7 2/3	1:43:53	5:45	18:01	18:47	19:33
8	1:48:24	6:00	18:48	19:36	20:24
8 1/3	1:52:55	6:15	19:35	20:25	21:15
8 2/3	1:57:26	6:30	20:22	21:14	22:06
9	2:01:57	6:45	21:09	22:03	22:57
9 1/3	2:06:28	7:00	21:56	22:52	23:48
9 2/3	2:10:59	7:15	22:43	23:41	24:39

MD *on-distance*		+Shin	+Prof	−WE	+RP	−Hurr
1/3	04:40	06	10	11	12	39
2/3	09:20	12	20	22	24	1:18
1	14:00	18	30	33	36	1:57
1 1/3	18:40	24	40	44	48	2:36
1 2/3	23:20	30	50	55	1:00	3:15
2	28:00	36	1:00	1:06	1:12	3:54
2 1/3	32:40	42	1:10	1:17	1:24	4:33
2 2/3	37:20	48	1:20	1:28	1:36	5:12
3	42:00	54	1:30	1:39	1:48	5:51
3 1/3	46:40	1:00	1:40	1:50	2:00	6:30
3 2/3	51:20	1:06	1:50	2:01	2:12	7:09
4	56:00	1:12	2:00	2:12	2:24	7:48
4 1/3	1:00:40	1:18	2:10	2:23	2:36	8:27
4 2/3	1:05:20	1:24	2:20	2:34	2:48	9:06
5	1:10:00	1:30	2:30	2:45	3:00	9:45
5 1/3	1:14:40	1:36	2:40	2:56	3:12	10:24
5 2/3	1:19:20	1:42	2:50	3:07	3:24	11:03
6	1:24:00	1:48	3:00	3:18	3:36	11:42
6 1/3	1:28:40	1:54	3:10	3:29	3:48	12:21
6 2/3	1:33:20	2:00	3:20	3:40	4:00	13:00
7	1:38:00	2:06	3:30	3:51	4:12	13:39
7 1/3	1:42:40	2:12	3:40	4:02	4:24	14:18
7 2/3	1:47:20	2:18	3:50	4:13	4:36	14:57
8	1:52:00	2:24	4:00	4:24	4:48	15:36
8 1/3	1:56:40	2:30	4:10	4:35	5:00	16:15
8 2/3	2:01:20	2:36	4:20	4:46	5:12	16:54
9	2:06:00	2:42	4:30	4:57	5:24	17:33
9 1/3	2:10:40	2:48	4:40	5:08	5:36	18:12
9 2/3	2:15:20	2:54	4:50	5:19	5:48	18:51

Course length picks out the row from the table on the right. Time offsets are small signed offsets from the expected elapsed time so only small values need be taken from the table on the left. Note that the separate components of the handicaps don't necessarily sort in the same order so columns may not directly match left and right.

By aligning the columns, distributing the sign into the table body, fixing the distance and specifying a range of signed time offsets we can add the two parts together to build a more specific table.

MD $\mp$*on-time*		Shin	Prof	WE	RP	Hurr
$-1^2/_3$	−22:35	−1:15	−4:15	+4:05	−3:55	+3:55
$-1^1/_3$	−18:04	−1:00	−3:24	+3:16	−3:08	+3:08
−1	−13:33	−45	−2:33	+2:27	−2:21	+2:21
$-^2/_3$	−09:02	−30	−1:42	+1:38	−1:34	+1:34
$-^1/_3$	−04:31	−15	−51	+49	−47	+47
	0	0	0	0	0	0
$+^1/_3$	+04:31	+15	+51	−49	+47	−47
$+^2/_3$	+09:02	+30	+1:42	−1:38	+1:34	−1:34
+1	+13:33	+45	+2:33	−2:27	+2:21	−2:21
$+1^1/_3$	+18:04	+1:00	+3:24	−3:16	+3:08	−3:08
$+1^2/_3$	+22:35	+1:15	+4:15	−4:05	+3:55	−3:55
+2	+27:06	+1:30	+5:06	−4:54	+4:42	−4:42
$+2^1/_3$	+31:37	+1:45	+5:57	−5:43	+5:29	−5:29
$+2^2/_3$	+36:08	+2:00	+6:48	−6:32	+6:16	−6:16
+3	+40:39	+2:15	+7:39	−7:21	+7:03	−7:03
$+3^1/_3$	+45:10	+2:30	+8:30	−8:10	+7:50	−7:50
$+3^2/_3$	+49:41	+2:45	+9:21	−8:59	+8:37	−8:37

MD 5=*distance*		Shin	Prof	WE	RP	Hurr
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
5	1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45

We simply add the corresponding entries to get the time-on-time-and-distance table specialized to this course length. The leftmost column serves no purpose but we do need to annotate the table to specify what course length it is effective for.

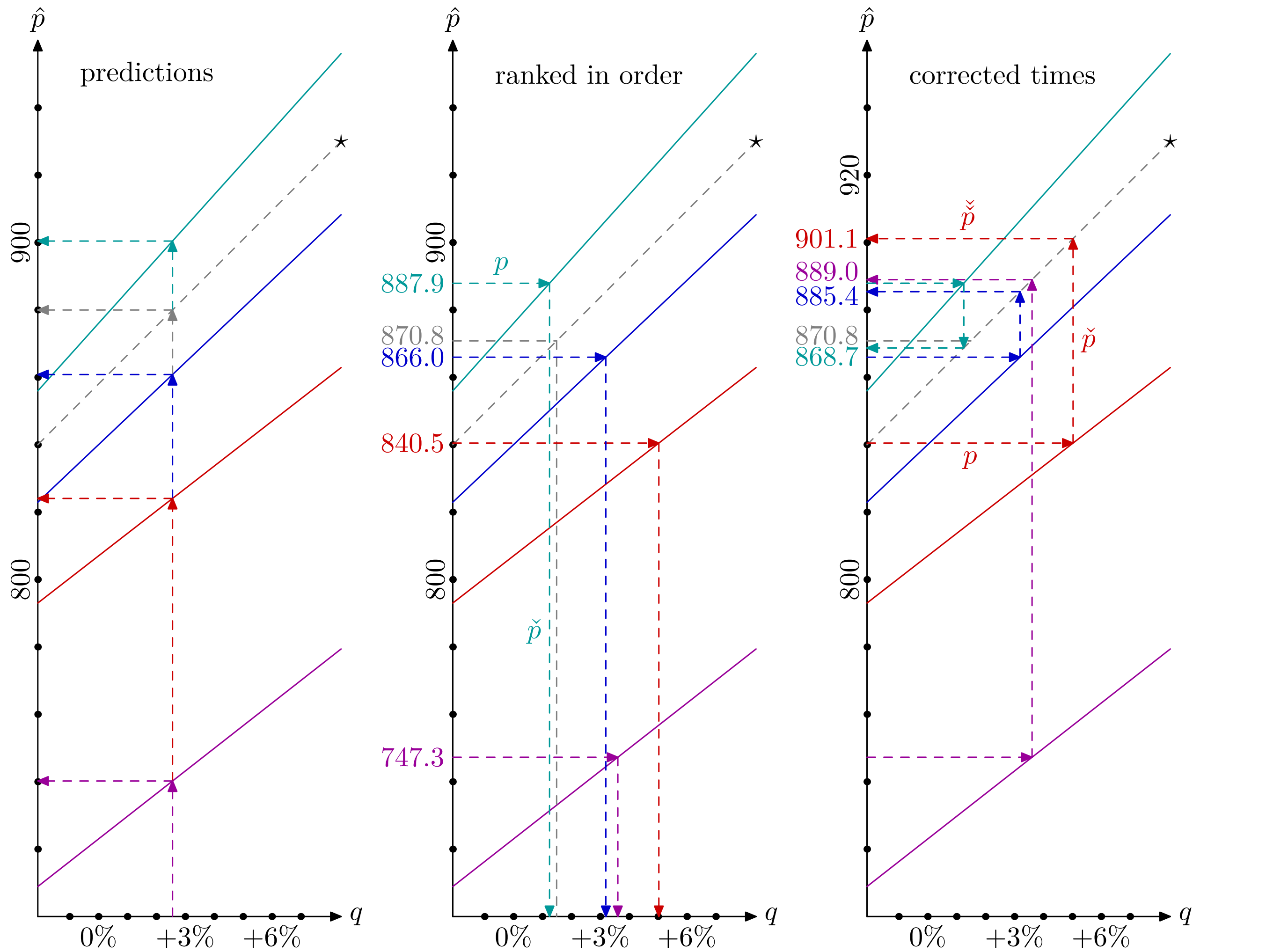
MD 5	Shin	Prof	WE	RP	Hurr
47:25	+0:15	−1:45	+1:20	−55	−5:50
51:56	+0:30	−54	+31	−08	−6:37
56:27	+0:45	−03	−18	+39	−7:24
1:00:58	+1:00	+48	−1:07	+1:26	−8:11
1:05:29	+1:15	+1:39	−1:56	+2:13	−8:58
1:10:00	+1:30	+2:30	−2:45	+3:00	−9:45
1:14:31	+1:45	+3:21	−3:34	+3:47	−10:32
1:19:02	+2:00	+4:12	−4:23	+4:34	−11:19
1:23:33	+2:15	+5:03	−5:12	+5:21	−12:06
1:28:04	+2:30	+5:54	−6:01	+6:08	−12:53
1:32:35	+2:45	+6:45	−6:50	+6:55	−13:40
1:37:06	+3:00	+7:36	−7:39	+7:42	−14:27
1:41:37	+3:15	+8:27	−8:28	+8:29	−15:14
1:46:08	+3:30	+9:18	−9:17	+9:16	−16:01
1:50:39	+3:45	+10:09	−10:06	+10:03	−16:48
1:55:10	+4:00	+11:00	−10:55	+10:50	−17:35
1:59:41	+4:15	+11:51	−11:44	+11:37	−18:22

We are expecting to finish this example 5 mile course in 1 h 10 min using the specialized table — but if the course is shortened we will have to fall back to the initial pair of tables. Note that columns entries in this table can flip sign — we can't abstract the sign of the column to the header as we did before.

†MD	Mechanical Drone	[13:33	14:00]
Shin	Shindig	[+45	+18]
Prof	Professor	[+2:33	+30]
WE	Winged Elephant	[−2:27	−33]
RP	Rhumb Punch	[+2:21	+36]
Hurr	Hurricane	[−2:21	−1:57]

4.5.3 An Example Time-on-Time-and-Distance Race with *Shindig* as Scratch

6.2 mile	$[k\ h]$	t (h:min:s→s)	p	$\check{q}$	$\check{\check{p}}$	$\check{\check{t}}$ (s→h:min:s)	$\Delta\check{\check{t}}$
Rhumb Punch	[954 876]	1:31:45→5505	887.9	+1.248%	868.7	5386→1:29:46	−13
Shindig	[858 858]	1:29:59→5399	870.8	+1.493%	870.8	5399→1:29:59	★
Mechanical Drone	[813 840]	1:29:29→5369	866.0	+3.194%	885.4	5490→1:31:30	+1:31
Hurricane	[672 723]	1:17:13→4633	747.3	+3.610%	889.0	5512→1:31:52	+1:53
Winged Elephant	[666 807]	1:27:01→5211	840.5	+5.028%	901.1	5587→1:33:07	+3:08

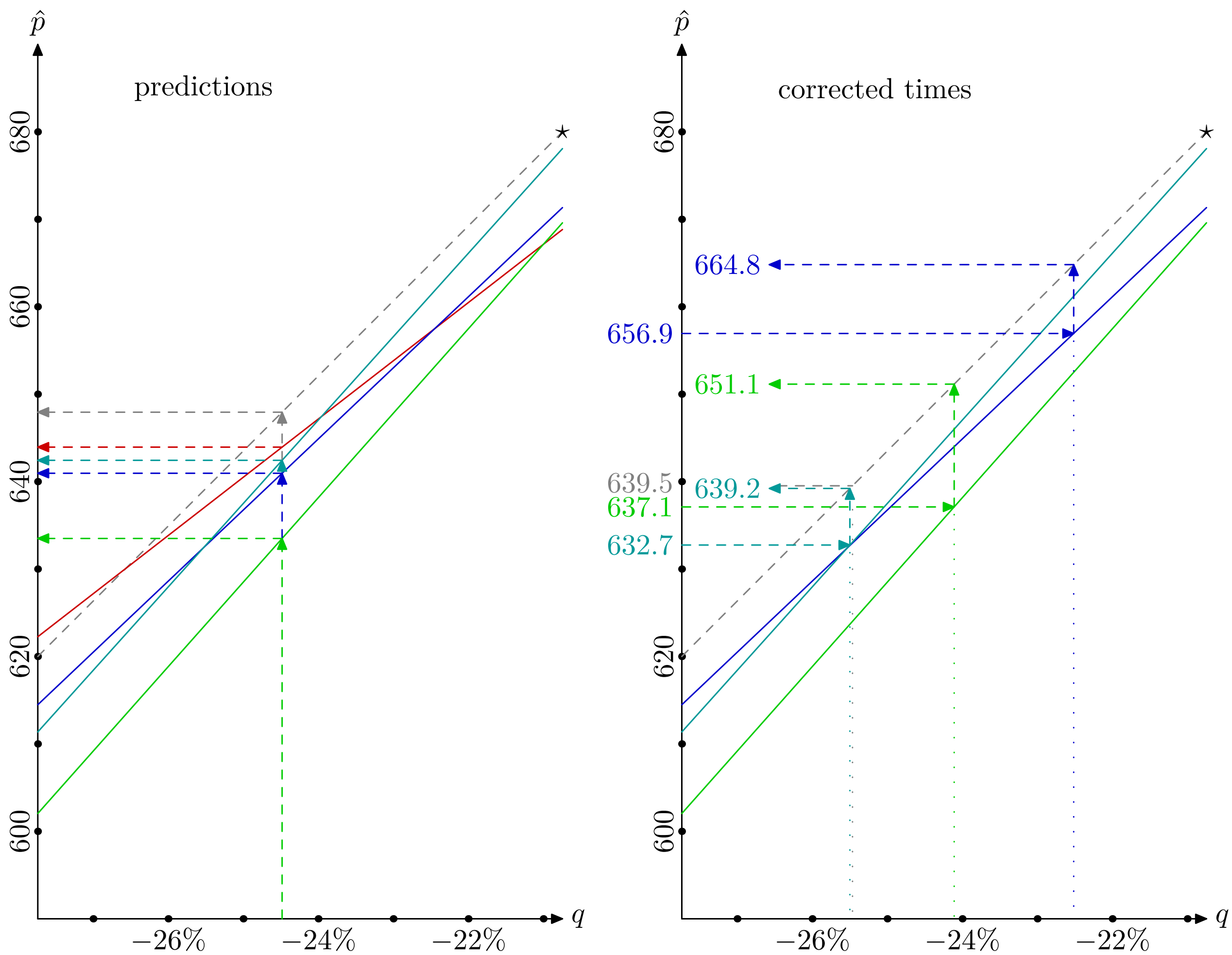


The left graph is just a blow-up of the graph of performance predictions for the five boats that raced, each boat as indicated by its colour. In this wind we are expecting ±20 s/mi., ±40 s/mi. and greater differences in pace between the boats — a problematic spread for single-factor handicapping schemes. Starting from the observed pace p on the vertical scale the middle graph reads off the value of $\check{q}$ on the horizontal scale — the boats are ranked in order along this axis. The right graph does the same for the observed pace p to the $\check{q}$ and then back to corrected pace $\check{\check{p}}$ on the vertical scale — here the values on the horizontal scale don't actually matter, only the lines count.

Note how the $\check{q}$ are tightly grouped on the q scale. The values are consistent with a medium wind and the handicapping between boats is appropriate for such a wind. If the race as whole was running 20% to 25% faster than average then the boats would be grouped much more tightly together (excluding *Hurricane*) and the time-on-time-and-distance handicapping would scale appropriately (i.e. the heavy-air part of the graph would kick in).

4.5.4 Another Example with More Wind

6.2 mile	$[k\ h]$	t (h:min:s→s)	p	$\check{q}$	$\check{\check{p}}$	$\check{\check{t}}$ (s→h:min:s)	$\Delta\check{\check{t}}$
Winged Elephant	[666 807]	1:01:09→3669	591.8	−32.32%	580.7	3601→1:00:01	−6:04
Rhumb Punch	[954 876]	1:05:23→3923	632.7	−25.50%	639.2	3963→1:06:03	−2
Shindig	[858 858]	1:06:05→3965	639.5	−25.46%	639.5	3965→1:06:05	★
Professor	[966 870]	1:05:50→3950	637.1	−24.11%	651.1	4037→1:07:17	+1:12
Mechanical Drone	[813 840]	1:07:53→4073	656.9	−22.52%	664.8	4122→1:08:42	+2:37



Winged Elephant had the heavy-air crew on board, won the start and got clean away on the first two legs. It's not a good heavy-air boat so this result is an outlier — we'd have expected it in the pack. Omitting its results lets us expand the scale and focus in on the remaining boats which were much more tightly grouped. Note how the expected differences in pace only cover 15 s/mile in total whereas in the previous example the expected differences were multiples of 20 s/mile over a much broader range. Nothing surprising happens on the right graph. What should be noted is that the differences in corrected times are very similar in the medium and heavy races, despite the boats being much more tightly clustered in the heavy-air race.

Chapter 5

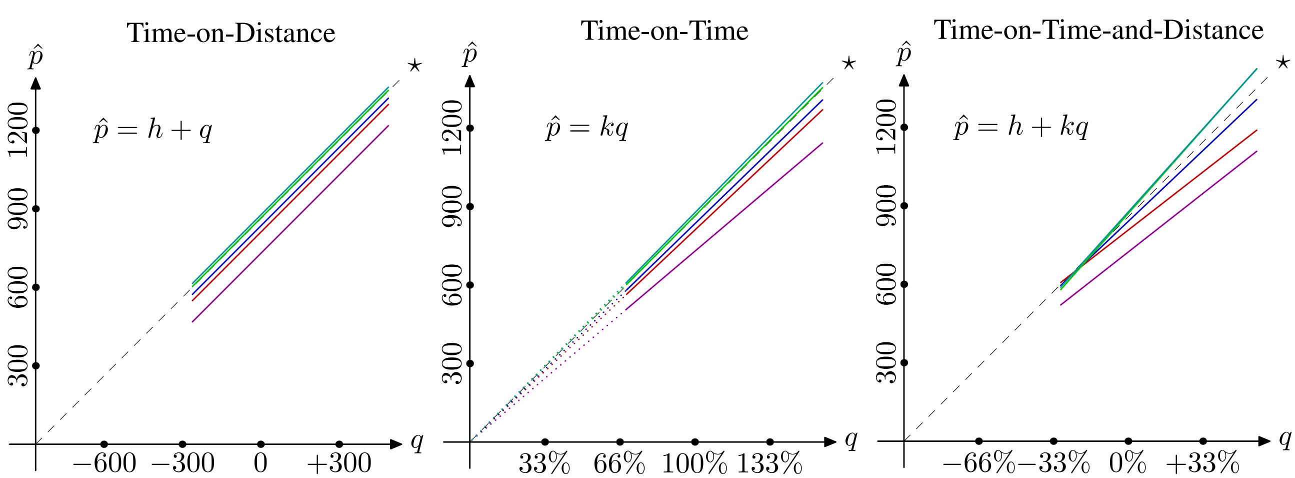
A Critique of Handicapping

5.1 Comparison of Different Styles of Handicapping

We can derive a corrected time formula for time-on-distance, time-on-time and performance curves from performance predictions in exactly the same manner as for time-on-time-and-distance using a mapping $q \mapsto \hat{p}$ where the interpretation of the variable q — and in particular $\check{q}$ as the preïmage of the observed p — is specific to each style of handicapping

$$
\begin{array}{llll}
q \mapsto h + q & \Longrightarrow \quad \check{\check{p}} = {}^{\star}h + \check{q}, & \check{q} = p - h & \text{(time-on-distance with handicaps } h \text{ and } {}^{\star}h\text{)} \\
q \mapsto k{\cdot}q & \Longrightarrow \quad \check{\check{p}} = {}^{\star}k{\cdot}\check{q}, & \check{q} = \dfrac{p}{k} & \text{(time-on-time with handicaps } k \text{ and } {}^{\star}k\text{)} \\
q \mapsto h + k{\cdot}q & \Longrightarrow \quad \check{\check{p}} = {}^{\star}h + {}^{\star}k{\cdot}\check{q}, & \check{q} = \dfrac{p-h}{k} & \text{(with handicaps } \begin{bmatrix} k & h \end{bmatrix} \text{ and } \begin{bmatrix} {}^{\star}k & {}^{\star}h \end{bmatrix}\text{)} \\
q \mapsto f(q) & \Longrightarrow \quad \check{\check{p}} = {}^{\star}f(\check{q}), & \check{q} = f^{\text{-}1}(p) & \text{(with performance curves } f \text{ and } {}^{\star}f\text{)}
\end{array}
$$

No matter the style of handicapping, the q is a free variable which models how slowly an individual race runs. As such it can be eliminated from a comparison of boats to create time allowances or interpreted as a preïmage of the observed pace in a corrected time formula to rank finishes. In either case, the free variable q is implicit in any handicapping application — and graphing that relationship, particularly in relation to the simple time-on-distance and time-on-time, makes clear how handicapping changes in response to the wind.



The origin point on these graphs (their lower left-hand corner) is associated with an inaccessible speed (a pace of $0\,\text{s/mile}$ is movement with infinite speed) and is the point of convergence for different boats using the time-on-time model (like a vanishing point in projective geometry). In any other context this point would usually be excluded from the displayed range of values.

Also notice how the q variable takes on a completely different interpretation in the three different styles on handicapping — nonetheless it is still easy to compare and contrast styles. All three styles model boats with straight lines — we haven't included an example of performance curves in this graphical comparison. In the time-on-distance case the lines are parallel, in the time-on-time case the lines converge and in the time-on-time-and-distance the convergence of different lines depends on the handicaps themselves.

5.2 The Pathetic Arithmetic of Today's Handicapped Racing

This is not to say that all existing handicapping systems can be replaced with a style of handicapping as described above, rather that they are already described as such even if that isn't made explicit in their formulation. But, measurement and handicapping authorities have, by various missteps, downplayed or otherwise obscured the performance prediction inherent to their handicaps:

By Having Inconveniently Gauged Handicaps

Any rescaling of the q-axis on one of the example graphs (i.e. any order preserving transformation of the q variable and substitution thereby) would still lead to the exact same time allowances and corrected times for the boats graphed. Any continuous *gauge transformation* is possible but, if we ignore performance curves, simple linear transformations are the only ones we need take an interest in. Each different choice of gauge leads to different suite of published handicaps and a different interpretation for the q variable while maintaining exactly the same form for the mapping $q \mapsto \hat{p}$.

And just as there is more than one way to express a straight line in algebraic geometry, there are other ways to parametrize a time-on-time-and-distance handicap for different seeming formulas that, nevertheless, yield identical corrected times. We have carefully parametrized our handicaps to be an obvious predictor of performance and therefore the most convenient for competitors when calculating time allowances.

By Having Mapped the Handicapping Relationship the Wrong Way Around

The straight lines on the three graph shown above are easily parametrized in a very natural way. For time-on-time-and-distance the typical parametrization would be, having identified the *ordinate axis* on the graph, to read off a slope and an intercept. It is important for a competitor to able compare itself simultaneously to all its competitors via time allowances, something the predictive mapping $q \mapsto \hat{p}$ makes easy. If instead the handicapping authority were to have flipped the axes on their graphs and parametrized the straight line of a handicap via the inverse mapping $p \mapsto \check{q}$ then calculating time allowances become cumbersome for competitors. The only practical way to do this in a fleet of boats is to manually reparametrize every handicap into the other more amenable form — the algebra involved is very straightforward but the calculation must be repeated for each and every boat — realistically a competitor would need to do this beforehand. In other words, each competitor would need to do the exact same reparametrization for every boat on the scratch sheet to generate data that the handicapping authority should have published as the handicaps in the first place.

The needless work this entails is not unique to time-on-time-and-distance handicaps. Time-on-time has a slightly simpler variation of the same problem in the TCF where you need to invert yours and all your competitors TCF's before going on the race course.

By Having Expressed Performance Predictions Relative to a Standard Boat

A VPP will model the performance of a given boat in units of *metres per second*, *knots* or *seconds per nautical mile*. While it is conceptually valid to build a system of units based on a standard boat rather than in absolute units it does present difficulties in interpretation absent the standard boat — a metre stick is a lot easier to carry around. A dubious argument for a relative gauge handicap is that, in racing, you don't need to know how well you are doing in absolute terms, you only need to know how well you are doing relative to your competition. But expressing performance predictions in absolute units in no way complicates that.

Performance handicapping authorities have traditionally based their handicaps on relative performance data which lacked an absolute reference frame but, given just a single VPP profile (most conveniently for the standard boat itself) could now reconstruct absolute handicaps for the entire fleet. Modern measurement handicaps which target club racing start with an absolute performance model and downgrade it to a relative one for no good reason.

Note that changing the choice of standard boat can be thought of as either as a unit conversion problem or as a gauge transformation of the numerical components of a handicap — the interpretation is a little different but operationally they are the same.

By Having Promulgated an Incomplete Corrected Time Formula

In the above corrected time formulation the scratch handicap appears explicitly as it should in any analysis of handicapping. Because the choice of scratch handicap makes no difference in how boats are ranked, handicapping authorities have often preëmptively chosen a scratch boat and have promulgated the resulting (and possibly simplified thereby) corrected time formula. This is an inexcusable mistake now that race results are always computerized. When the scratch boat isn't remotely representative of division to which it applies — with a predetermined and arbitrary choice of scratch boat this must occur more often than not — corrected times become difficult to interpret. In the ideal situation, each competitor could recalculate corrected times with their own boat as scratch — with web published results this is actually quite easy to achieve.

Also note that the choice of the standard boat with which to gauge a relative performance handicap is completely independent of the choice of scratch boat with which to express corrected times. Conflating these two is another misjudgment often made by handicapping and organizing authorities — expressing handicaps in absolute units sidesteps this particular misfeature.

A related confusion is a belief that gauge transformations alter corrected times while preserving the ranking of boats. This is not so — changing the choice of scratch boat in the corrected time formulation is the only way to effect corrected times. Nevertheless, for a published corrected time formula with an implicit scratch boat a subsequent gauge transformation would also implicitly and inadvertently select a different boat as scratch. This can lead to misunderstandings and is best avoided. Never having let such a crippled corrected time formula be used would have been the best course of action.

By a Lack of Adequate Documentation

Its not unusual for corrected time itself to be left undefined in the class rules or other core constitutional documents of a handicapping authority. This is a glaring omission given that the *Racing Rules of Sailing* also leaves the definition of corrected time undefined. Most handicap racing is taking place with no formal rules as to how to rank finishers. Given that the measurement or handicapping authority, the one and only class association for handicapped racing, through omission fails to empower boats to race legitimately, it should be no surprise that they fail to document how to interpret their handicaps.

And all racers in handicap racing should have a basic understanding of time allowances and how they are used. Yet no formally recognized documents for these basic tools of handicapped racing has ever existed. Why? Why don't class rules demand that time allowance tables be made available to competitors by race organizers? Why is handicapped racing surrounded by this amorphous cloud of ignorance? Perhaps the *Racing Rules of Sailing* should allow for a single handicapped class association to be served by independent handicapping authorities — this would allow a single organization to finally proselytize and enforce best practices.

Part III

The Arithmetic of Applying Handicaps

Chapter 6

Introduction

Handicapping is the art of prediction. Having predicted how a boat will perform we can account for it on the race course. At its simplest, how each boat performs relative to its prediction can be used either to directly rank finishers or, equivalently, to calculate corrected times which are then ranked. Corrected times provide context but require an arbitrary choice of a common scratch boat.

6.1 Different Styles of Handicapping

6.1.1 Corrected Times

Corrected times are usually one's first introduction to the handicapping of sailboats. Before widespread computerization, simple corrected time formulae had been promulgated by handicapping authorities to allow for hand computation by race committees and easy verification of results by competitors. This practice continues to this day but without the prior justification. The algebraic simplification that leads to easy-to-apply formulae disguises the underlying predictive nature of the handicaps and rips them of the context needed to easily understand what a corrected time actual represents.

A prediction of elapsed times suitable for placing boats in a race will require exactly one degree of freedom, a free variable to measure of how slow or fast a race has progressed. By unifying this free variable between our own and our competitors' boats we can compare the observed elapsed time of our boat to the predicted elapsed times of the others (see section 7.6 Plotting the Critical Equations for Time Allowances). Likewise, by singling out a scratch boat and unifying pairwise between each boat and the scratch boat we can map the observed elapsed time of each boat to its corrected time, a prediction of the elapsed time it should have finished with were it the same as the scratch boat (see section 7.1 Defining Equations for Corrected Time.). This is much easier to explain using algebra as in the following chapter 7 On Distance, On Time or On Time and Distance.

The handicap for a boat (or more properly the handicapping relationship between its one free variable and the predicted elapsed time) can be arbitrarily complex, parametrized by a multiple factors as with *performance curve scoring*, or very simple, consisting of only a single factor as with *time-on-distance* and *time-on-time* handicapping. The simplest styles still account for the vast majority of handicap racing. But single-factor handicaps are very limited in the precision available to describe the performance of a boat, only being capable of describing a boat's performance on average and unable to represent how a boat's performance varies across a range of wind speeds.

Note that corrected time formulae do not directly expose the handicapping relationship that maps a free variable to a predicted elapsed time. All this information is erased from the algebraic expression of the formula.

6.1.2 Oversimplification of Time-on-Distance and Time-on-Time Styles

Also, single-factor handicaps allow us to circumvent the one degree of freedom inherent to the handicapping relationship and simply treat the handicap as a singular prediction of pace. Performance relative to that prediction, via a difference for time-on-distance or via a ratio for time-on-time can be used to rank finishers and to calculate corrected times. But treating a single-factor handicap as a prediction with no degrees of freedom has the downside of completely robbing it of the context needed to draw a comparison between different styles of handicapping; it thoroughly obscures any understanding of how this prediction can be used in different winds. Note that algebraically nothing changes from the general understanding as described above, via unification over the single degree of freedom inherent to a handicapping prediction; only our interpretation of these algebraic operations differ. In the case of a single-factor no-degree-of-freedom handicapping the algebra is magical, erasing information which would otherwise makes it impractical to use these handicaps across different wind ranges. But magic does not help with understanding.

6.1.3 Developing Our Understanding in This and the Following Chapter

Handicapping authorities have embraced magical algebra and have not bothered to justify their handicapping choices in any comprehensible way. Time-on-distance versus time-on-time was the only choice generally available until IMS introduced *performance curve scoring*. Performance curve scoring has a misleading name owing to its initial and quite peculiar avoidance of using corrected times — this *scoring technique* is, of course, a style of handicapping wholly amenable for use with corrected times as the *Racing Rules of Sailing* mandates.

Fuzzy thinking has not been restricted to club racers but has extended all the way to the top of handicapping authorities, despite their being able to promulgate perfectly functional handicapping techniques. Not surprisingly, documentation has been abysmal.

We will develop our understanding slowly in the following chapter, from the bottom up, starting with the historical corrected time formulae and then progressing through time allowances, the plotting of critical elapsed times through to the general formulation in terms of pace. Note that we will do so in a thoroughly modern context, preferring absolute gauge handicaps which express typical performance in seconds per mile — handicaps which should express expected performance around a normal race course with average winds.

6.1.4 Time-on-Distance versus Time-on-Time: A Synthesis

Time-on-distance versus time-on-time is a false dichotomy based on a historical bias. Either describe single-factor handicapping schemes that map elapsed to corrected times using simple proportions. But both are just special cases of time-on-time-and-distance, a two-factor synthesis that has much greater predictive power than either of the single-factor styles of handicapping as it will automatically respond to changes in wind speed.

Time-on-distance is the easiest for competitors to grasp on the race course, especially when shortening the course isn't an option. Applying time-on-time on the course is a little different but still straightforward with the handicaps discussed here. When shortening isn't an option, time-on-time-and-distance is just as easy to use as time-on-time as precomputed tables can replace the need for mental arithmetic.

6.1.5 Performance Curve Scoring: A Further Generalization

Handicapping with performance curves can be considered a nonlinear extension of time-on-time-and-distance handicapping and, as such, requires pre-computed tables of time allowances for competitors to be able to gauge their progress throughout a race. Also, despite the name, performance curve scoring is not a *scoring technique* but rather a *style of handicapping*.

Performance curves are named after the graphical representation of a boat's predicted performance over a range of wind speeds. As originally introduced the horizontal axis of a performance curve was always wind speed but this requirement is too strict for our purposes — we will have to wait until end of the following chapter (see section 7.7 Performance Curves) for the formalism needed to properly describe all styles of handicapping.

6.2 Common Conventions for Handicaps

6.2.1 Definitions and Variable Name Conventions for Handicaps

We may use a boat's typical course-average pace in seconds per mile as either a time-on-distance handicap or a time-on-time handicap. In the algebraic expression of the handicapping relationship for time-on-distance this single factor handicap is denoted the *distance coefficient* h or simply the *time-on-distance factor* h. Whereas in the algebraic expression of the handicapping relationship for time-on-time we denote this the *time coefficient* k or the *time-on-time factor* k. This handicap, the typical course-average pace for a boat, shall be denoted by the variable h or k as befits its usage but when we want to emphasize the common origin of both we will refer to it as a *general-purpose handicap* (abbreviated GPH) and may denote it g. We wont deal with the implications of using the same handicap $g = h = k$ for both time-on-distance and time-on-time handicapping until the next chapter.

A two-factor time-on-time-and-distance handicap $\begin{bmatrix} k & h \end{bmatrix}$ consists of a time coefficient k and a distance coefficient h both measured in seconds per nautical mile — the h factor is the boat's expected course-average pace in normal conditions and the k is the difference from that pace as the wind decreases — remember that a slower pace is represented by a greater number of seconds per mile. When we expect a race to run 10% slower than normal we would expect the course-average pace for a particular boat to be $h + 10\%k$ based upon its two-factor handicap.

We indicate the scratch boat for a fleet with a star ★ and one of ${}^{\star}h$, ${}^{\star}k$ or $\begin{bmatrix} {}^{\star}k & {}^{\star}h \end{bmatrix}$ for its handicap.

Here handicaps are a reckoning of a boat's expected performance in absolute units (i.e. $^{\mathrm{s}}/_{\mathrm{mile}}$) but it would have been possible to gauge performance relative to a standard boat. In that case we would have unitless time coefficients k as a percentage of the standard and distance coefficients h as offsets from it. It is easier to understand corrected time relationships in terms of handicaps which gauge absolute performance so we will put off exploring such relative gauge handicaps for now.

6.2.2 Example Boats and their Performance Handicaps

From a small anonymized data-set (not included in this document) we have regressed some two-factor time-on-time-and-distance handicaps $\begin{bmatrix} k & h \end{bmatrix}$ as well as single-factor general-purpose handicaps g. These handicaps are rounded to the closest multiple of $3\,^{\mathrm{s}}/_{\mathrm{mile}}$. Table entries $\begin{bmatrix} k & h \end{bmatrix}$ and g in units seconds per mile ($^{\mathrm{s}}/_{\mathrm{mile}}$) are duplicated as minutes and seconds per mile ($^{\mathrm{min:s}}/_{\mathrm{mile}}$) in the adjacent parenthesized expressions.

Example Boat	$[k\ h]$	g	Make
Hurricane	[672(11:12) 723(12:03)]	729(12:09)	Buddy 24
Winged Elephant	[666(11:06) 807(13:27)]	810(13:30)	Frequency 24
Mechanical Drone	[813(13:33) 840(14:00)]	834(13:54)	See in Sea 30
Shindig	[858(14:18) 858(14:18)]	861(14:21)	Raider 28
Professor	[966(16:06) 870(14:30)]	864(14:24)	Stone 22
Rhumb Punch	[954(16:54) 876(14:36)]	876(14:36)	Chimera 33

The fitted single-factor g handicap is similar but not identical to the distance coefficient h of the corresponding time-on-time-and-distance handicap. For a handicap derived from a velocity prediction programme, the distance coefficient of a time-on-time-and-distance handicap, the single factor of a time-on-time handicap and the single factor of a time-on-distance handicap would all be identical. But for a performance handicap there can be small differences in these factors, especially for a small sample set. Unless each boat has performance data from races that match the overall distribution of wind conditions we should expect small discrepancies between the distance coefficient of a fitted time-on-time-and-distance handicap and the single-factor general-purpose handicap. What is more, there are three perfectly reasonable but quite different ways to regress a general-purpose handicap from a data-set, each of which yields subtly different results.

Note that handicapping authorities and race organizers never express handicaps in minutes and seconds per mile despite this being the most convenient form for a competitor to use on the water. When developing time allowance tables it is always necessary to convert handicaps into this preferred form.

Chapter 7

On Distance, On Time or On Time and Distance

7.1 Defining Equations for Corrected Time

Time-on-Distance

$$\check{\check{t}} - {}^{\bigstar}hd = t - hd$$

Time-on-Time

$$\frac{\check{\check{t}}}{{}^{\bigstar}k} = \frac{t}{k}$$

Time-on-Time-and-Distance

$$\frac{\check{\check{t}} - {}^{\bigstar}hd}{{}^{\bigstar}k} = \frac{t - hd}{k}$$

Given the boat's actual elapsed time t, the corrected time $\check{\check{t}}$ is how we would have expected it to finish were it the same type as the scratch boat $\bigstar$ (as best as the style of handicapping allows). For any boat that handicaps as scratch its corrected time will be identical to its own elapsed time — such a boat can compare a hypothetical elapsed time it might have achieved directly against the actual corrected times of other boats to delineate how it might have placed differently.

In essence, we rank boats according to the fiction that this is a one-design race of scratch boats. Elapsed times gives us a measurement of performance; via our performance prediction model we can determine the corresponding performance we would have expected were this boat of the scratch type; these are our *corrected* measurements of performance with which we can rank competitors. The fiction that this is a one-design race of scratch boats makes it easy to see how the race would have come out differently if those boats that happen to handicap as scratch were to have different elapsed times — all this is against the backdrop of the other boats with different handicaps which can't be reshuffled so easily — and should only a single boat handicap as scratch then this isn't a particularly useful observation.

The choice of the scratch handicap must be common to all the boats in a division but is otherwise unconstrained — selecting a scratch boat within each division gives a meaningful context to corrected times.

7.2 Calculating Corrected Times Using Formulae

7.2.1 Formulae for Mapping via an Intermediate Commensurable $\check{u}$

We place boats in a race by solving the defining equations for corrected time $\check{\check{t}}$. This is best understood as a two step process.

on Distance	on Time	on Time and Distance
$\check{\check{t}} - {}^{\bigstar}hd = \check{u} = t - hd$	$\dfrac{\check{\check{t}}}{{}^{\bigstar}k} = \check{u} = \dfrac{t}{k}$	$\dfrac{\check{\check{t}} - {}^{\bigstar}hd}{{}^{\bigstar}k} = \check{u} = \dfrac{t - hd}{k}$

We measure how well each boat performs relative to its own handicap — the intermediate $\check{u}$ term (with one check on top) — then map that back through the scratch $\bigstar$ prediction to get corrected times $\check{\check{t}}$ (with two checks on top)

$$
\begin{array}{llll}
\check{\check{t}} = \check{u} + {}^{\bigstar}hd & \text{where} & \check{u} = t - hd & \text{for time-on-distance} \\
\check{\check{t}} = {}^{\bigstar}k\check{u} & \text{where} & \check{u} = \dfrac{t}{k} & \text{for time-on-time} \\
\check{\check{t}} = {}^{\bigstar}k\check{u} + {}^{\bigstar}hd & \text{where} & \check{u} = \dfrac{t - hd}{k} & \text{for time-on-time-and-distance}
\end{array}
$$

We say the intermediate $\check{u}$ term is *commensurable* because it can be used to sort boats into their finishing places. Note that only in the time-on-distance case is $\check{u}$ measured in units of time. The mapping from the intermediate $\check{u}$ to the corrected times $\check{\check{t}}$ does not effect the ordering of entries so finishing places can be determined by:

- sorting on $\check{u}$ directly (not necessarily a time but orderable in its own units),
- sorting on the corrected times $\check{\check{t}}$ for a preselected scratch boat
- or sorting on corrected times calculated with regards to any scratch boat chosen after the fact

Because being the scratch boat makes interpreting one's own results so much easier, it is advantageous for each competing boat to be able to recalculate corrected times with itself as scratch. And because how boats place can be determined by the intermediate *commensurable* $\check{u}$ it is not strictly necessary to calculate corrected times at all; nevertheless, we always report corrected times $\check{\check{t}}$ to give a context to results.

7.2.2 Race Ties on Corrected Time When using a Rounding Rule

Our freedom to choose an arbitrary scratch boat after the fact vanishes when corrected times are defined as rounded to the nearest second. While the ordering of finishing places is mostly preserved by a mapping from the commensurable $\check{u}$ to rounded corrected times $[\check{\check{t}}]$ (enclosing a time in square brackets is used to denote the time rounded to the nearest second) there is a possibility to collapse finishes with sub-second differences in unrounded corrected time to finishes with equal rounded corrected time. What is more, which entries collapse down to a tied finishing place critically depend on the particular choice of scratch boat.

Before 2005, the rule book required corrected times to be rounded to the nearest second and some rating and handicapping class rules still require some kind of rounding. This hand computation bias is unfortunate and no longer relevant — it is actually easier to get consistently correct behaviour out of a computer programme without a rounding rule. The only drawback to calculating corrected times using exact arithmetic is that there is no consensus on how to report corrected times with sufficient precision to visibly break ties yet still be easy to read. Only reporting rounded corrected times keeps (rare) sub-second tie-breaking calculations hidden; representing corrected times as mixed fractions only visibly orders results to the second; and exact decimalization leads to repeating decimals which are easy to compare at a glance but overly long to display.

7.3 Time-on-Time-and-Distance As a Generalization

Time-on-time-and-distance handicapping is a straightforward generalization of time-on-distance and time-on-time handicapping. It can encompass either of the single-factor schemes. Should all the boats in a division have a two-factor handicap with a common time coefficient then time-on-time-and-distance handicapping becomes the same as time-on-distance. And should each boat have a two-factor handicap with its two coefficients equal to each other then time-on-time-and-distance handicapping becomes the same as time-on-time.

Historically, time-on-distance was widely used in North America and time-on-time elsewhere. Both compare actual elapsed time to predicted elapsed time (via subtraction or division, respectively) and can still be used to place boats when winds depart from the mean. Each single-factor handicapping scheme is tied to an implicit model which predicts how a boat's performance must change for a given change in wind speed as extrapolated solely from its average performance. Between any particular pair of boats, one or the other style of handicapping will model the relative changes in performance the better, but it is known that across a whole fleet time-on-time leads to more effective handicapping. The time-on-time-and-distance synthesis can explicitly model changes in performance per boat and has the potential to keep racing competitive for all boats in light, medium or heavy air.

7.4 Time Allowances on the Race Course

Although it is possible to work out corrected times throughout a race, competitors usually prefer to work with *time allowances* Δt (read as *delta t*) which would be the differences in elapsed time between pairs of boats that should correct out the same. When you determine the time allowance between your own boat and that of a competitor you can compare it to the actual lead in time to find out whether you would place ahead or behind. Determining time allowances for each or your competitors will tell how you compare to each and, through that, how you will place on the whole — we will develop a personalized *time allowance table* which will allow you to look up a row and then read off the time allowance for each of your competitors across the columns. The key to the row, as used for look-up, will depend on the style of handicapping, whether time-on-distance, time-on-time or time-on-time-and-distance. This row of time allowances doesn't give you a total ordering of all finishers but will let you determine how your own boat will place with less effort than tracking corrected times.

In this chapter the Δt notation will refer specifically to a time allowance. Note that Δt or a *delta* corresponding to a handicapping factor (Δh or Δk) are always signed quantities even though in the body of a table it may be possible to omit the sign and only display absolute values.

7.4.1 The Critical Proportionality and Making a Table of Allowances

Time-on-distance allowances are the easiest to work with as they do not change throughout the race; although, should we choose, they can calculated to any fraction of the race as a judgment on our progress up to that point. $\Delta t = \Delta h d$ (read as *"delta h" times "d"*) will be the time allowance between you and your competitor for Δh the corresponding difference in time-on-distance handicaps. With the interval δ as a conveniently sized fraction of the typical race course (e.g. $\delta = 1\,\text{mi}$ or $\delta = {}^1\!/\!_3\,\text{mi}$) we can tabulate given the Δh with respect to each of our competitors

archetype for time-on-distance

Distance	$\cdots$	Time Allowance	$\cdots$
δ		$\Delta h \delta$	
2δ		$2\Delta h \delta$	
3δ		$3\Delta h \delta$	
4δ		$4\Delta h \delta$	
5δ		$5\Delta h \delta$	
$\vdots$		$\vdots$	

So the table rows vary over distance and there needs be a separate column for each different possible absolute value of Δh; we will allow a positive Δh and a negative Δh with the same magnitude to collapse to the same table column. Reading the row corresponding to the course distance you can determine, for each competitor, how far ahead in time you need to be. A negative Δh occurs when your competitor has the faster boat and your negative time allowance will be time in your favour.

Note that differences in handicap pop up in the 1 mile row and need not be specifically labelled although we have added ruled lines around that row to aid lookup. Also note that the table headings as shown above are obvious from the context; in typical usage we would instead label the columns with the make of boat or even, in a small and diverse fleet, the names of competitors. And even though it is typical for handicapping authorities to state handicaps in hundreds of seconds per mile, time allowances in hundreds or thousands of seconds would be inconvenient. Time allowances are more often stated in hours, minutes and seconds.

For time-on-time and time-on-time-and-distance handicapping, time allowances between you and your competitors will increase proportionately throughout the race

$$\begin{aligned} \Delta t : \Delta k \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t : k \times 1\,\mathrm{mi} && \text{for time-on-time} \\ \Delta t - \Delta hd : \Delta k \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t - hd : k \times 1\,\mathrm{mi} && \text{for time-on-time-and-distance} \end{aligned}$$

Δt will be the time allowance between you and your competitor, Δk or $\begin{bmatrix}\Delta k & \Delta h\end{bmatrix}$ the corresponding difference in handicapping factors, t will be your elapsed time and k or $\begin{bmatrix}k & h\end{bmatrix}$ your handicap. By tracking time in intervals of $k \times 1\,\mathrm{mi}$ you get proportions suitable for mental arithmetic.

Making a table of time-on-time-and-distance allowances from this proportionality is easy. For every $k \times 1\,\mathrm{mi}$ in excess of hd that you spend on the course, the base allowance of Δhd goes up by $\Delta k \times 1\,\mathrm{mi}$. Multiplying the right-hand side of the ratios by 1 mi is the same as stripping *per-mile* from the units — this balances the units on both sides of the ratio and generally makes everything easier to deal with — it does however make the algebraic presentation of the proportionality uglier and seemingly cluttered. Note that we never specify units in the body of a table. And a table of allowances for time-on-time handicapping is even simpler to make as it does not depend on course length. With $\delta = 1\,\mathrm{mi}$ we have tables for time-on-time-and-distance and time-on-time, respectively

archetype for time-on-time-and-distance

	Elapsed Time	···	Time Allowance	···
⋮	⋮		⋮	
-2δ	$hd - 2k\delta$		$\Delta hd - 2\Delta k\delta$	
-1δ	$hd - k\delta$		$\Delta hd - \Delta k\delta$	
	hd		Δhd	
$+1\delta$	$hd + k\delta$		$\Delta hd + \Delta k\delta$	
$+2\delta$	$hd + 2k\delta$		$\Delta hd + 2\Delta k\delta$	
⋮	⋮		⋮	
table only holds for course length d				

archetype for time-on-time

	Elapsed Time	···	Time Allowance	···
δ	$k\delta$		$\Delta k\delta$	
2δ	$2k\delta$		$2\Delta k\delta$	
3δ	$3k\delta$		$3\Delta k\delta$	
4δ	$4k\delta$		$4\Delta k\delta$	
5δ	$5k\delta$		$5\Delta k\delta$	
6δ	$6k\delta$		$6\Delta k\delta$	
7δ	$7k\delta$		$7\Delta k\delta$	
8δ	$8k\delta$		$8\Delta k\delta$	
⋮	⋮		⋮	

To use the table read off your elapsed time from the first column (ignoring the greyed out column as displayed here) to identify the row then read off the time allowance from appropriate column on the right. It may be necessary to interpolate between rows. Note that the parameter $\delta = 1\,\mathrm{mi}$ in the archetypical tables above not only serves to strip per-mile from the units of the k and Δk but gives us the option of reducing the increment between rows by a factor of three via an interval $\delta = {}^{1}\!/_{3}\,\mathrm{mi}$ for a finer-grained table with less need for interpolation. In an actual table there would be no units and no parameter, simply a chosen interval between rows and numeric values with implicit units.

A particular time allowance table will be specialized to your own handicap and the deltas (the differences between their handicap and ours) from your competitors. The time-on-time case is like all tables based on a single handicapping factor as the whole column will be positive or negative depending on the sign of the handicap delta and, as such, the table needs only display the absolute value of the time allowance in the body of the table. For time-on-time-and-distance table of a given course length the sign of a time allowance could cross zero as elapsed time increases — this makes it necessary to display the sign of the time allowance in the table body itself. To handle a shortened course in a time-on-time-and-distance race we would need two tables, one for the time coefficient of the handicap and one for the distance coefficient, then add the two time allowance terms as needed — this is much more general but also less convenient when sailing a course of a known length. And while it would be atypical to include sign of the term in the body of either of these two tables, not being needed in the table body itself, it might be more readable to display these terms as signed entries.

time-on-time-and-distance table 2 of 2
archetype for the time coefficient

±Excess	Time	⋯	±Allowance	⋯
δ	$k\delta$		$\Delta k\delta$	
2δ	$2k\delta$		$2\Delta k\delta$	
3δ	$3k\delta$		$3\Delta k\delta$	
4δ	$4k\delta$		$4\Delta k\delta$	
5δ	$5k\delta$		$5\Delta k\delta$	
⋮	⋮		⋮	

time-on-time-and-distance table 1 of 2
archetype for the distance coefficient

Distance	Base Time	⋯	Base Allowance	⋯
β	$h\beta$		$\Delta h\beta$	
2β	$2h\beta$		$2\Delta h\beta$	
3β	$3h\beta$		$3\Delta h\beta$	
4β	$4h\beta$		$4\Delta h\beta$	
5β	$5h\beta$		$5\Delta h\beta$	
⋮	⋮		⋮	

Note that we have written the time coefficient table (table 2 of 2) to the left of the distance coefficient table (table 1 of 2). This is backwards. We will do a lookup right-to-left; the right table gives us the base from which we offset with the excess from the left table. This reversal of the natural left-to-right ordering (and with respect to the algebraic terms in the archetypal time-on-time-and-distance table) is for consistency with the handicaps themselves which are *time-on-time-and-distance* with the time coefficient listed before and to the left of the distance coefficient.

Also note that the range and interval between rows need not be related between these two tables. In these mock-ups we have used the two independent parameters β and δ for the intervals to make that clear; even though, in practice, these would both be the same — i.e. either a a coarsely-grained $\beta = \delta = 1\,\mathrm{mi}$ or a finely-grained $\beta = \delta = 1/3\,\mathrm{mi}$. For expository purposes we have shown both these tables with two initial columns which in the context of single-factor handicapping would be used for *on-distance* and the other for *on-time* lookup. We should specialize these tables by omitting the first column of the time coefficient table (table 2 of 2) as this table only requires *on-time* lookup.

A dual-purpose table suitable for both *on-distance* and *on-time* lookup would be all that is needed for a general-purpose handicap; this uses the same handicap for both time-on-distance and time-on-time racing. Such a dual-purpose table also gives expected elapsed times for an average race of a given distance — such information can be useful on the water. Unless we know otherwise this would be the typical form for a single-factor time allowance table in whatever context it is used.

archetype for a dual-purpose table

Distance	Elapsed Time	⋯	Time Allowance	⋯
δ	$g\delta$		$\Delta g\delta$	
2δ	$2g\delta$		$2\Delta g\delta$	
3δ	$3g\delta$		$3\Delta g\delta$	
4δ	$4g\delta$		$4\Delta g\delta$	
5δ	$5g\delta$		$5\Delta g\delta$	
6δ	$6g\delta$		$6\Delta g\delta$	
7δ	$7g\delta$		$7\Delta g\delta$	
⋮	⋮		⋮	

7.4.2 Handicap Deltas for Time-on-Time-and-Distance

Here are some time-on-time-and-distance performance handicaps to use in worked examples. The three columns of differences $\begin{bmatrix}\Delta k & \Delta h\end{bmatrix}$ are calculated from the perspective of the boat marked by a ★. Not to say that your boat need be the scratch boat for the purposes of reporting corrected times, but simply that it is convenient to use the ★ notation for this purpose as well. Units are s/mi.

Boat	Handicap	Handicap Differences			Make
Hurricane	[672 723]	★	[−141 −117]	[−186 −135]	Buddy 24
Winged Elephant	[666 807]	[−6 +84]	[−147 −33]	[−192 −51]	Frequency 24
Mechanical Drone	[813 840]	[+141 +117]	★	[−45 −18]	See in Sea 30
Shindig	[858 858]	[+186 +135]	[+45 +18]	★	Raider 28
Professor	[966 870]	[+294 +147]	[+153 +30]	[+108 +12]	Stone 22
Rhumb Punch	[954 876]	[+282 +153]	[+141 +36]	[+96 +18]	Chimera 33

(see subsection 10.1.8 for the same in a relative gauge)

7.4.3 Example of Time-on-Time-and-Distance Table of Allowances for *Hurricane*

A single table for time-on-time-and-distance can only apply to a course of fixed length. Let's assume $d = 4\,\text{mi}$. Your boat *Hurricane* has a handicap of $\begin{bmatrix}k & h\end{bmatrix} = \begin{bmatrix}672\,\text{s/mi} & 723\,\text{s/mi}\end{bmatrix}$. Stripping *per-mile* from the units gives us $\begin{bmatrix}672\,\text{s} & 723\,\text{s}\end{bmatrix} = \begin{bmatrix}11\,\text{min}\,12\,\text{s} & 12\,\text{min}\,3\,\text{s}\end{bmatrix}$. Multiplying h by course distance d gives you an *expected* finish of 48 min 12 s. You can easily calculate time allowances for all the boats you are racing against. For each competitor's handicap delta $\begin{bmatrix}\Delta k & \Delta h\end{bmatrix}$ add $\Delta k \times 1\,\text{mi}$ to their base allowance of $\Delta h \times 4\,\text{mi}$ every 11 min 12 s in excess of 48 min 12 s elapsed. If we finish before 48 min 12 s we just flip the sign of the excess so that for every 11 min 12 s before 48 min 12 s elapsed we subtract the competitors $\Delta k \times 1\,\text{mi}$ from their base allowance $\Delta h \times 4\,\text{mi}$. The expected hd and interval k are only dependent on your own handicap and the course distance making it easy to track all your competitors simultaneously with only a little bit of preparation.

★ Hurricane 4	[672 723]	Winged E. [−6 +84]	Mech. D. [+141 +117]	Shindig [+186 +135]	Professor [+294 +147]	Rhumb P. [+282 +153]
−2	0:25:48	+5:48	+3:06	+2:48	0:00	+0:48
−1	0:37:00	+5:42	+5:27	+5:54	+4:54	+5:30
	0:48:12	+5:36	+7:48	+9:00	+9:48	+10:12
+1	0:59:24	+5:30	+10:09	+12:06	+14:42	+14:54
+2	1:10:36	+5:24	+12:30	+15:12	+19:36	+19:36
+3	1:21:48	+5:18	+14:51	+18:18	+24:30	+24:18

Their boat *Winged E.* with a $\begin{bmatrix}666\,\text{s/mi} & 807\,\text{s/mi}\end{bmatrix}$ handicap differs by $\begin{bmatrix}\Delta k & \Delta h\end{bmatrix} = \begin{bmatrix}-6\,\text{s/mi} & +84\,\text{s/mi}\end{bmatrix}$ from your boat *Hurricane*. Using the distance coefficient of the handicap delta and a course length of 4 mi we calculate $\Delta h \times 4\,\text{mi}$ which resolves to +84 s/mi times 4 mi (four times +1 min 24 s) yielding a base allowance of +5 min 36 s; this number (+5:36) can be read from the highlighted base row of the table in the *Winged E.* column. Then, using the time coefficient delta of −6 s, for every 11 min 12 s in excess of 48 min 12 s of elapsed time the time allowance of +5 min 36 s is reduced by 6 s. If you finish with an elapsed time of 1 hour we can look-up the best row at 59 min 24 s of elapsed time to estimate that you need to win by 5 min 30 s. For a fast finish of 37 min you need to win by 5 min 42 s.

Note that the greyed out initial column, included for expository purposes, is not used for look-up and might best be omitted from the table.

7.4.4 The Same *Four-Mile* Table of Allowances for *Hurricane* but Finer-Grained

This next table advances by thirds of $\Delta k \times 1/3\,\mathrm{mi}$ every $k \times 1/3\,\mathrm{mi} = 3\,\mathrm{min}\,44\,\mathrm{s}$ to make interpolation easier. Even with the added rows, for large Δk interpolation can still be a bit tricky.

To compensate for the added height we have, in addition to restricting the range, abbreviated the column headings and split out a separate legend. This can sometimes aid legibility, although in this case it hardly makes a difference. Also in the legend we are using minutes and seconds per mile rather than just seconds per mile for handicaps to be consistent with the body of the main table.

Hurricane 4		WE	MD	Shin	Prof	RP
−2/3	0:40:44	+5:40	+6:14	+6:56	+6:32	+7:04
−1/3	0:44:28	+5:38	+7:01	+7:58	+8:10	+8:38
	0:48:12	+5:36	+7:48	+9:00	+9:48	+10:12
+1/3	0:51:56	+5:34	+8:35	+10:02	+11:26	+11:46
+2/3	0:55:40	+5:32	+9:22	+11:04	+13:04	+13:20
+1	0:59:24	+5:30	+10:09	+12:06	+14:42	+14:54

★	Hurricane	[11:12 12:03]
WE	Winged Elephant	[−6 +1:24]
MD	Mechanical Drone	[+2:21 +1:57]
Shin	Shindig	[+3:06 +2:15]
Prof	Professor	[+4:54 +2:27]
RP	Rhumb Punch	[+4:42 +2:33]

Their boat *Shindig* with a $\left[858\,\mathrm{s/mi} \quad 858\,\mathrm{s/mi}\right]$ handicap differs by $\left[\Delta k \quad \Delta h\right] = \left[+186\,\mathrm{s/mi} \quad +135\,\mathrm{s/mi}\right]$. The time allowance of $+135\,\mathrm{s/mi} \times 4\,\mathrm{mi} = +9\,\mathrm{min}$ at 0:48:12 on the elapsed time clock is increased by $+186\,\mathrm{s} = +3\,\mathrm{min}\,6\,\mathrm{s}$ every $11\,\mathrm{min}\,12\,\mathrm{s}$. By thirds that would be $+1\,\mathrm{min}\,2\,\mathrm{s}$ every $3\,\mathrm{min}\,44\,\mathrm{s}$. $\Delta k \times 1/3\,\mathrm{mi}$ is still quite large so it might me useful to note that $+3\,\mathrm{min}\,6\,\mathrm{s}$ every $11\,\mathrm{min}\,12\,\mathrm{s}$ can be rounded to $+3\,\mathrm{min}$ every $11\,\mathrm{min}$ and then scaled down to $+3\,\mathrm{s}$ every $11\,\mathrm{s}$. If you finish with an elapsed time of 1 hour then you need to win by greater than $12\,\mathrm{min}\,6\,\mathrm{s}$ — say $12\,\mathrm{min}\,15\,\mathrm{s}$. For a fast finish of $37\,\mathrm{min}$ you need to win by $5\,\mathrm{min}\,54\,\mathrm{s}$.

7.4.5 More Examples of Time-on-Time-and-Distance Tables of Allowances

We can then compare tables from two different perspectives side-by-side. First we will identify the boats and their handicaps. These examples also make it clear why we have to include the sign of the time allowance in the body of the table for a fixed course length time-on-time-and-distance table of allowances.

Example of Time Allowances from *Shindig's* Perspective Your boat *Shindig* has a handicap of $\left[858\,\mathrm{s/mi} \quad 858\,\mathrm{s/mi}\right]$. In units stripped of *per-mile* we have $858\,\mathrm{s}$ which is $14\,\mathrm{min}\,18\,\mathrm{s}$. The course is $4\,\mathrm{mi}$. From the distance coefficient $14\,\mathrm{min}\,18\,\mathrm{s} \times 4 = 57\,\mathrm{min}\,12\,\mathrm{s}$. From the time coefficient in increments of a third $14\,\mathrm{min}\,18\,\mathrm{s} \times 1/3 = 4\,\mathrm{min}\,46\,\mathrm{s}$. The time allowance for each competitor will be $\Delta h \times 4\,\mathrm{mi}$ at $57\,\mathrm{min}\,12\,\mathrm{s}$ of elapsed time (0:57:12) increasing by $\Delta k \times 1/3\,\mathrm{mi}$ every $4\,\mathrm{min}\,46\,\mathrm{s}$.

Second Perspective for Time-on-Time-and-Distance Time Allowances Your boat *Mechanical Drone* has a handicap of $\left[813\,\mathrm{s/mi} \quad 840\,\mathrm{s/mi}\right]$. $\left[813\,\mathrm{s} \quad 840\,\mathrm{s}\right]$ is $\left[13\,\mathrm{min}\,33\,\mathrm{s} \quad 14\,\mathrm{min}\,0\,\mathrm{s}\right]$. The course is $4\,\mathrm{mi}$. $14\,\mathrm{min} \times 4 = 56\,\mathrm{min}$. In increments of a third $13\,\mathrm{min}\,33\,\mathrm{s} \times 1/3 = 4\,\mathrm{min}\,31\,\mathrm{s}$. The time allowance for each competitor will be $\Delta h \times 4\,\mathrm{mi}$ at $56\,\mathrm{min}$ elapsed (0:56:00) increasing by $\Delta k \times 1/3\,\mathrm{mi}$ every $4\,\mathrm{min}\,31\,\mathrm{s}$.

Example Tables of Time Allowances for Time-on-Time-and-Distance on a 4 mi Course

★ Shindig 4 [858 858]		Mech. D. [−45 −18]	Professor [+108 +12]	Rhumb P. [+96 +18]	Winged E. [−192 −51]	Hurricane [−186 −135]
$-1\frac{1}{3}$	0:38:08	−0:12	−1:36	−0:56	+0:52	−4:52
-1	0:42:54	−0:27	−1:00	−0:24	−0:12	−5:54
$-\frac{2}{3}$	0:47:40	−0:42	−0:24	+0:08	−1:16	−6:56
$-\frac{1}{3}$	0:52:26	−0:57	+0:12	+0:40	−2:20	−7:58
	0:57:12	−1:12	+0:48	+1:12	−3:24	−9:00
$+\frac{1}{3}$	1:01:58	−1:27	+1:24	+1:44	−4:28	−10:02
$+\frac{2}{3}$	1:06:44	−1:42	+2:00	+2:16	−5:32	−11:04
$+1$	1:11:30	−1:57	+2:36	+2:48	−6:36	−12:06
$+1\frac{1}{3}$	1:16:16	−2:12	+3:12	+3:20	−7:40	−13:08
$+1\frac{2}{3}$	1:21:02	−2:27	+3:48	+3:52	−8:44	−14:10

★ Mech. D. 4 [813 840]		Shindig [+45 +18]	Professor [+153 +30]	Rhumb P. [+141 +36]	Winged E. [−147 −33]	Hurricane [−141 −117]
$-1\frac{1}{3}$	0:37:56	+0:12	−1:24	−0:44	+1:04	−4:40
-1	0:42:27	+0:27	−0:33	+0:03	+0:15	−5:27
$-\frac{2}{3}$	0:46:58	+0:42	+0:18	+0:50	−0:34	−6:14
$-\frac{1}{3}$	0:51:29	+0:57	+1:09	+1:37	−1:23	−7:01
	0:56:00	+1:12	+2:00	+2:24	−2:12	−7:48
$+\frac{1}{3}$	1:00:31	+1:27	+2:51	+3:11	−3:01	−8:35
$+\frac{2}{3}$	1:05:02	+1:42	+3:42	+3:58	−3:50	−9:22
$+1$	1:09:33	+1:57	+4:33	+4:45	−4:39	−10:09
$+1\frac{1}{3}$	1:14:04	+2:12	+5:24	+5:32	−5:28	−10:56
$+1\frac{2}{3}$	1:18:35	+2:27	+6:15	+6:19	−6:17	−11:43

(see 10.1.10 for the same in a relative gauge)

7.5 And Distance Allowances on the Race Course

There is an another way to consider the allowances between boats which, while less quantifiable than time allowances, can be useful to keep in mind. We can reinterpret the handicaps for time-on-time as distance-on-distance. For distance-on-distance, rather than requiring yourself to be a certain time ahead of your competitor, you require yourself to be ahead in distance. And like time-on-distance these allowances do not change throughout the race. On a light air day or a heavy air day you still need to be the same distance ahead at the finish. These distance allowances can be made mathematically precise by modeling boats as having a constant pace throughout a race — which highlights a glaring weakness — when winds are significantly different at the finish line than other parts of the race then time allowances and distance allowances become out of sync, making them useless for our purposes.

Much as for the single-factor time-on-time, we can reinterpret the handicaps for two-factor ime-on-time-and-distance as distance-and-time-on-distance. For what they are worth, distance-and-time-on-distance allowances are just as easy to understand as distance-on-distance allowances. After accounting for the time-on-distance part of the allowance you then take the distance-on-distance part into account. We'll make this precise in a later chapter when we discuss pursuit races.

7.6 Plotting the Critical Equations for Time Allowances

7.6.1 Via Equations for Critical Elapsed Times $\hat{t}$

On the race course we always use proportions to track time allowances Δt but it can still be useful to plot time allowances or the accompanying *critical elapsed times* $\hat{t}$ (your elapsed time plus the time allowance) (in general, we use the hat notation to distinguish a prediction from an actual observation — we need the context to know that $\hat{t}$ here refers to a critical elapsed time). We will calculate critical elapsed times and time allowances from the perspective of your boat identified by the ★ notation. So the unadorned h, k or $\begin{bmatrix} k & h \end{bmatrix}$ will refer to your competitor's boat and the handicaps with the $\bigstar$ superscript will refer to your boat. This usage is analogous but not the same as that used for defining corrected times — that is, we identify one boat as special and refer each other boat to it in turn. Here we will take an elapsed time t for your boat $\bigstar$ and predict how a competitor's boat should finish $\hat{t}$ (with a hat on top) based only upon the your and their handicap.

Time-on-Distance

$$\hat{t} - hd = t - {}^{\bigstar}hd$$

Time-on-Time

$$\frac{\hat{t}}{k} = \frac{t}{{}^{\bigstar}k}$$

Time-on-Time-and-Distance

$$\frac{\hat{t} - hd}{k} = \frac{t - {}^{\bigstar}hd}{{}^{\bigstar}k}$$

From this we can calculate the time allowance $\Delta t = \hat{t} - t$. Notice that the mapping from the observed t to the predicted $\hat{t}$ happens in reverse to how corrected times are defined — a further mapping of the $\hat{t}$ to a corrected time with your competitor's boat supplying the handicapping correction and your boat as scratch would result in the original t — precisely what we require for a time allowance.

7.6.2 Via Critical Proportions

And we can plot time allowances without recourse to critical elapsed times using equations derived from the critical proportions. We need to take care that the Δk is moved to the right-hand side of the equation to avoid a potential division by zero so we might as well do the same for the Δh. When Δk is zero the time-on-time-and-distance relationship degenerates to the time-on-distance equation.

on Distance

$$\Delta t = \Delta hd$$

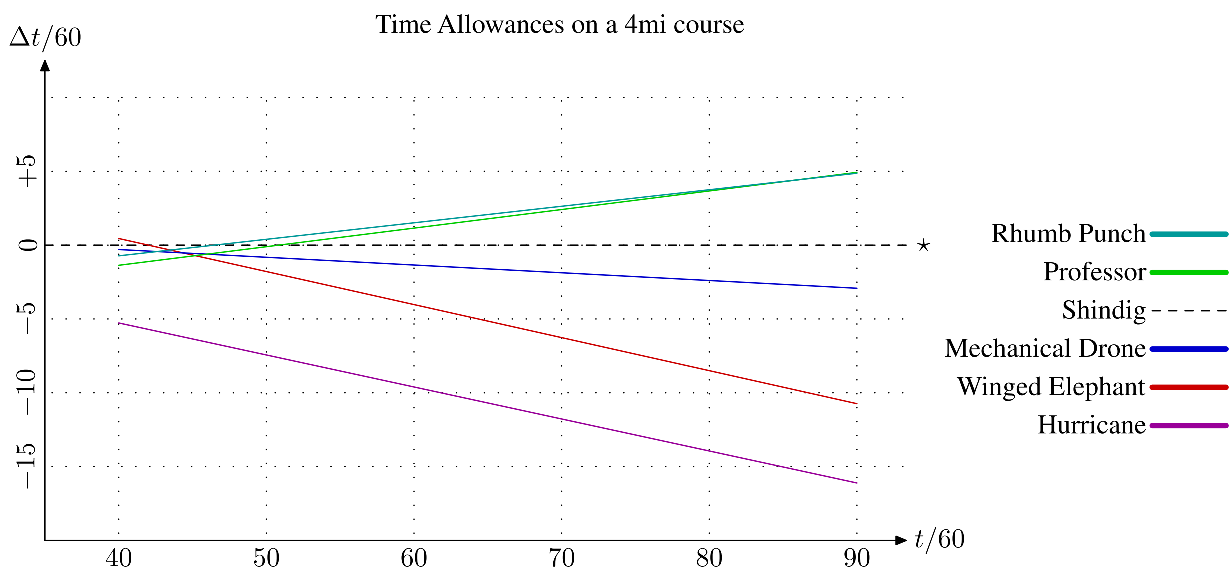
on Time

$$\Delta t = \Delta k \times \frac{t}{{}^{\bigstar}k}$$

on Time and Distance

$$\Delta t = \Delta hd + \Delta k \times \frac{t - {}^{\bigstar}hd}{{}^{\bigstar}k}$$

The time-on-distance relationship is plotted as horizontal parallel lines. For time-on-time the graphed lines will intersect at the origin, extrapolated off to the left hand side of the plot.



These are the time allowances Δt for each boat with respect to *Shindig* on a $d = 4\,\text{mi}$ course using time-on-time-and-distance handicapping. The axes are scaled in minutes. The same graph can be used on any length course just by stretching or contracting the scale on the axes in step with the change in length — the plotted lines will never change.

Plotting the critical elapsed times $\hat{t}$ directly instead of the Δt leads us to the study of performance curves. But first, as an aside...

7.6.3 Critical Elapsed Times and Time Allowances Revisited

There is another and more constructive alternative to using critical equations or critical proportionalities in the definition of the critical elapsed times and time allowances. For each boat, let $\hat{t}$ (with a hat) be a variable of predicted elapsed time defined with respect to a generalized free variable u correlated in sense (but not formally) with the progress of the race

$$\hat{t} = ku + hd$$

The mapping $u \mapsto \hat{t}$ is unimportant in of itself — all that matters is that each boat has a predicted elapsed time which can be compared for equal values of u. We'll identify one of the boats as *yours* ★ and let the Δt variable be the difference in $\hat{t}$ between each of *theirs* and your boat.

$$\Delta t = (ku + hd) - ({}^{\bigstar}ku + {}^{\bigstar}hd) = \Delta ku + \Delta hd$$

At elapsed time t constraining the u to satisfy $t = {}^{\bigstar}ku + {}^{\bigstar}hd$ will then define the $\hat{t}$ as a critical elapsed time and the Δt as a time allowance for each of their boats with respect to you. This description is more abstract and therefore not as immediately useful as our previous definitions, but it nicely demonstrates how well chosen the k and h handicapping factors are. And it is easy to see how this leads to the same formulae we have already introduced as well as to the building of time allowance tables.

7.7 Performance Curves

7.7.1 Corrected Times from Performance Lines

We have defined corrected times using simple formulae but for the most sophisticated style of handicapping, performance curve scoring, we need a better understanding. First we will work in the most natural form for handicapping, through course-average and corrected course-average paces rather than elapsed and corrected time

on Distance	on Time	on Time and Distance
$\check{\check{p}} - {}^{\bigstar}h = \check{q} = p - h$	$\dfrac{\check{\check{p}}}{{}^{\bigstar}k} = \check{q} = \dfrac{p}{k}$	$\dfrac{\check{\check{p}} - {}^{\bigstar}h}{{}^{\bigstar}k} = \check{q} = \dfrac{p - h}{k}$

Here we can see that the handicapping calculations map course-average pace p to the intermediate and commensurable $\check{q}$ (with a check on top) and then to the corrected pace $\check{\check{p}}$ (with two checks on top) via simple linear transformations parametrized over handicaps — course distance is factored out of the calculation so there are no extraneous variables. Note that the commensurable $\check{q}$ is not necessarily in units of pace whereas the course-average pace p and corrected pace $\check{\check{p}}$ both are.

Next we consider a boat's handicap to be a linear function on p which we will call *chk*: $p \mapsto \check{q}$ (*check* looking like an upside-down *hat* or *cap*) with an explicitly named inverse function *cap*: $\check{q} \mapsto p$ (*cap* being an un-*chk*)

$$\check{q} = \mathrm{chk}(p) \qquad \Longleftrightarrow \qquad p = \mathrm{cap}(\check{q})$$

so that the corrected time may be defined and calculated (via the corrected pace)

$$\mathrm{chk}^{\bigstar}(\check{\check{p}}) = \check{q} = \mathrm{chk}(p) \qquad \Longrightarrow \qquad \check{\check{p}} = \mathrm{cap}^{\bigstar}(\check{q}) = \mathrm{cap}^{\bigstar}(\mathrm{chk}(p))$$

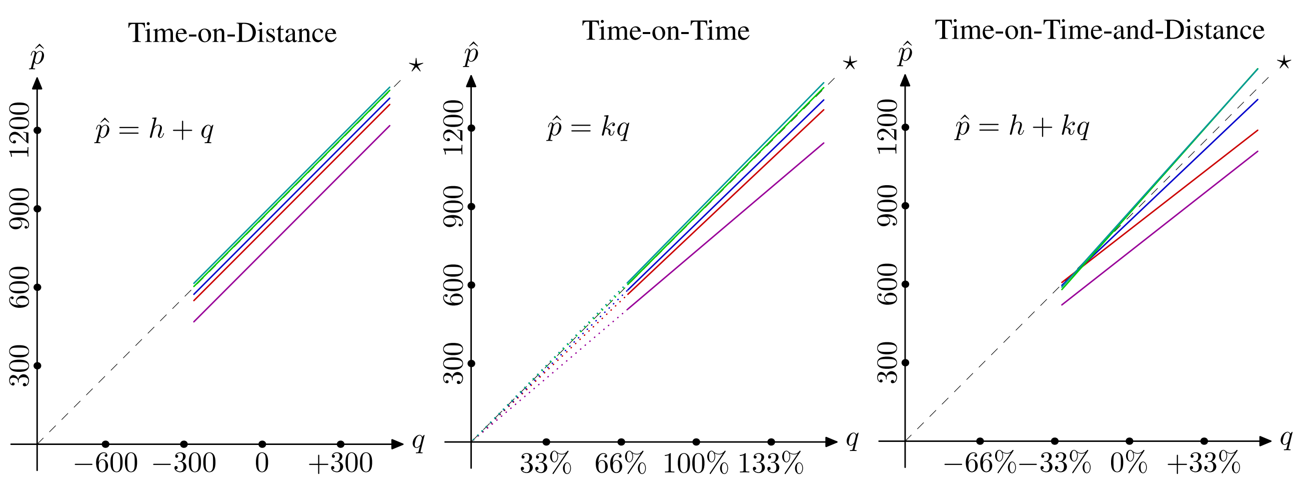
This is the most general definition of corrected time and is universally applicable. Note that we consider each boat to have its own *chk* and *cap* functions which correspond to its own handicap. The form the function takes is tightly constrained by the style of handicapping, with the handicapping factors acting as parameters to completely specify the relationship. There is no definition of corrected time which cannot be interpreted in this manner.

For the handicapping already described we have

$$\begin{aligned}
\mathrm{chk}(p) &= p - h &&\iff && \mathrm{cap}(q) = h + q && \text{for time-on-distance}\\
\mathrm{chk}(p) &= \frac{p}{k} &&\iff && \mathrm{cap}(q) = kq && \text{for time-on-time}\\
\mathrm{chk}(p) &= \frac{p-h}{k} &&\iff && \mathrm{cap}(q) = h + kq && \text{for time-on-time-and-distance}
\end{aligned}$$

For sensibly defined handicaps the *cap* function has the simpler representation, with the *chk* function most conveniently defined as its inverse. Finally we note that, despite the order in which we introduced them, the *cap* function is the more fundamental of the two functions — *cap*: $q \mapsto \hat{p}$ returns our predicted pace $\hat{p}$ (with a hat or *cap* on top to distinguish it from an observed pace) for a generalized parameter q correlated to the wind speed. We have already seen the q for time-on-time-and-distance handicapping as the percentage of time slower than the average we expect for a race. The domain of q has a different interpretation for each style of handicapping and, as we will see later, for different handicapping gauges. Despite its importance we wont develop a standard nomenclature to refer to this variable or its domain — we will most often refer to $\check{q}$ as simply "q-check" but given the fundamental nature of the *cap* function we may also refer to $\check{q}$ as the *pace preïmage* without ambiguity.

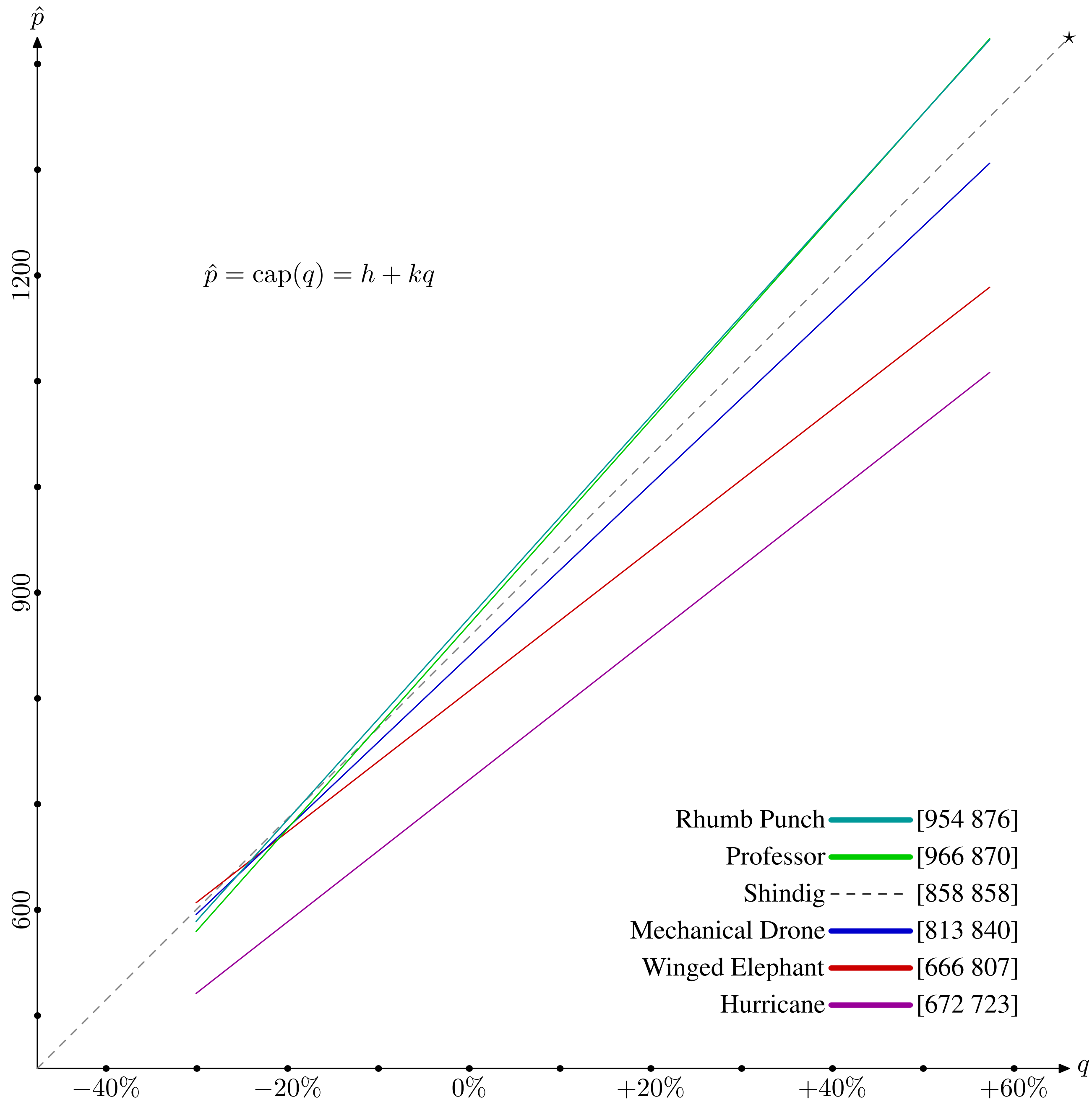
7.7.2 Examples of Performance Lines for Different Styles of Handicapping



The graphed straight lines are the handicap as realized by the *cap* function. Here we compare a slower cyan green boat which is good in heavy air with a middling blue boat and a faster red boat that is best in light air. The other magenta purple boat is fastest of all (although its planing potential isn't shown here at all). Note that the origin of the graph corresponds to an infinite speed so is best omitted from the x and y scale — we have included it here for comparison purposes only, to highlight the convergence of the time-on-time performance lines at the origin.

Allowing curved lines as handicaps would give us even more predictive power at the cost of greater complexity. Allowing discontinuous lines would let us model the behaviour of planing boats but would be useless as a handicap as the *chk* function would be ill-defined at the discontinuity — a performance curve with sharp kinks would be difficult to analyze but serve perfectly well as a handicap.

7.7.3 A Hi-Res Example of Time-on-Time-and-Distance Performance Lines



7.7.4 The Critical Equations of Course-Average Pace Allowances

Just as for the defining equations of corrected time we can also reformulate the critical equations of time allowances in terms of the *critical course-average pace* $\hat{p}$ for each of your competitors and your own observed course-average pace p as identified by the ★

$$\hat{p} - h = p - {}^{\bigstar}h \qquad\qquad \frac{\hat{p}}{k} = \frac{p}{{}^{\bigstar}k} \qquad\qquad \frac{\hat{p} - h}{k} = \frac{p - {}^{\bigstar}h}{{}^{\bigstar}k}$$

The unadorned *chk* and *cap* will refer to your competitor's boat and the functions with the ★ superscript will refer to your boat so that the critical course-average pace may be characterized and calculated

$$\mathrm{chk}(\hat{p}) = \mathrm{chk}^{\bigstar}(p) \qquad \Longrightarrow \qquad \hat{p} = \mathrm{cap}(\mathrm{chk}^{\bigstar}(p))$$

and course-average pace allowance defined

$$\Delta p = \hat{p} - p \qquad \Longrightarrow \qquad \Delta p = \mathrm{cap}(\mathrm{chk}^{\bigstar}(p)) - p$$

7.7.5 An Aside on Course-Average Pace Allowances

A more symmetric way to define the pace allowance would be to let the Δp be a variable which describes the difference in the *cap* function of a pair of boats (theirs and yours with yours identified by a ★) and which varies according to the free variable q

$$\Delta p = \mathrm{cap}(q) - \mathrm{cap}^{\bigstar}(q)$$

So that when we constrain the q such that

$$t = \mathrm{cap}^{\bigstar}(q) \times d$$

then Δp will be the course-average pace allowance between theirs and yours at your time t. This skips the definition of the critical course-average pace but clearly follows from the same argument.

The distribution law of multiplication allows us to collect the terms in the free variable q so that $\Delta p = \Delta k q$ when the observed course-average pace $t/d = p = {}^{\bigstar}kq$ for time-on-time and $\Delta p = \Delta kq + \Delta h$ when the observed course-average pace $t/d = p = {}^{\bigstar}kq + {}^{\bigstar}h$ for time-on-time-and-distance. It then becomes easy to eliminate the q, either in a proportionality

$$\begin{aligned} \Delta p : \Delta k &\overset{\text{in proportion}}{=} p : {}^{\bigstar}k && \text{for time-on-time} \\ \Delta p - \Delta h : \Delta k &\overset{\text{in proportion}}{=} p - {}^{\bigstar}h : {}^{\bigstar}k && \text{for time-on-time-and-distance} \end{aligned}$$

Or in an equation when $\Delta k \neq 0$

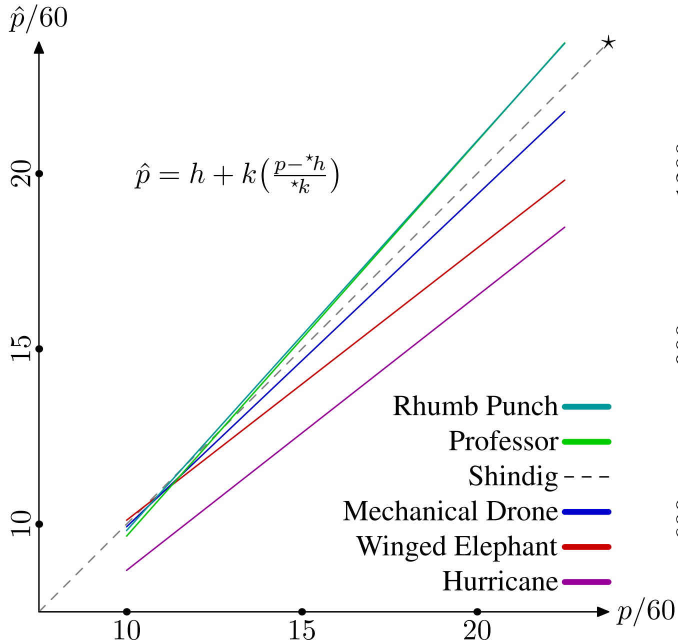
$$\frac{\Delta p}{\Delta k} = q = \frac{p}{{}^{\bigstar}k} \qquad\qquad \frac{\Delta p - \Delta h}{\Delta k} = q = \frac{p - {}^{\bigstar}h}{{}^{\bigstar}k}$$

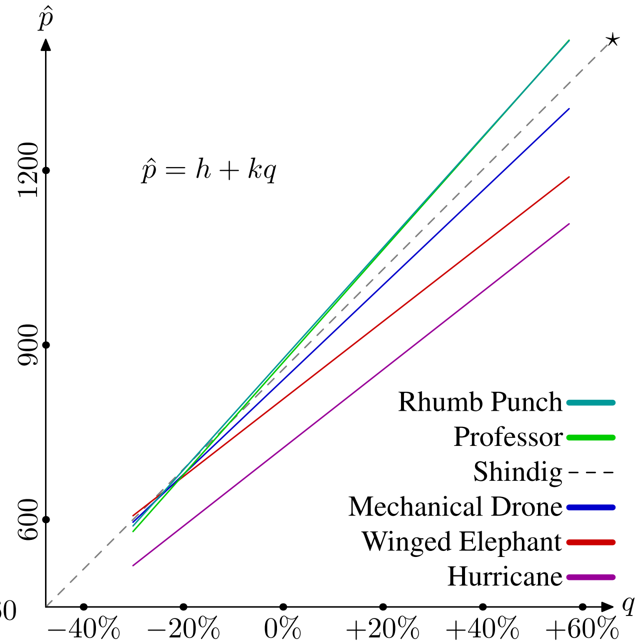
When $\Delta k = 0$ time-on-time becomes level racing and time-on-time-and-distance degenerates to simple time-on-distance. For time-on-distance Δp is fixed at Δh whatever the time $t/d = p$ might be.

7.7.6 Burying the Critical Course-Average Pace

It would be tempting but ill-advised to interpret the $\hat{p} = \mathrm{cap}(\mathrm{chk}^{\bigstar}(p))$ as a *critical instantaneous pace* for your own *instantaneous pace* or a *critical leg-average pace* for your own *leg-average pace*. The handicaps which determine the $\hat{p}$ are whole race predictions integrating over fast and slow legs. In practice *time allowances* are more useful than *pace allowances*.

A plot of the critical $\hat{p}$ from your p will look identical to a plot of performance lines where the scale and axes were chosen so that the your own boat's performance line lies on the main diagonal — only the scale on the x-axis will differ between the two plots.





Critical course-average paces aren't terribly interesting in of themselves — the plot of the more simply defined performance lines shows exactly the same relationships between boats. We used the same trick of simplifying the scale on the x-axis to equate the two plots as we did when we tabulated the table of time allowances using only additions and trivial multiplications via the proportionality, but had to do so at less than elegant intervals of elapsed times. A table with rows at elapsed times of 5 min, 10 min, 15 min and so on corresponds to the the more complex plot on the left — a table that would be far more challenging to produce using only mental arithmetic.

7.7.7 Linearizing the Plot of Performance Curves

If we allow performance curves and their representative *cap* functions instead of straight performance lines then the correspondence between the plot of predicted paces and that of performance curves becomes more interesting. Morphing the x-axis (with a nonlinear graticule based on the *cap*$^{\bigstar}$ function) makes the plots equivalent by straightening out the $\bigstar$ boat's curve onto the main diagonal, and should largely straighten out the curves of competing boats. This is most relevant when each *cap* function is derived from a VPP which will map wind pace p_{wind} to boat pace $\hat{p}$ on a convex curve. Straighter lines are easier to interpolate and overlapping straight lines are easier to distinguish than overlapping curves. If performance curves are to be visualized it makes the most sense to plot them this way.

7.7.8 Performance Curves from a VPP

We now have the proper framework for (true) performance curve scoring. At a given wind speed v_{wind} we can derive a polar from the VPP and integrate over the expected points of sail to generate an overall predicted pace $\hat{p}$. This gives us $\hat{p}$ as a function of wind. Let the wind strength be numerically represented by wind pace p_{wind} so that the *cap*: $p_{\text{wind}} \mapsto \hat{p}$ is a nice increasing function. On the other hand, an *cap*: $v_{\text{wind}} \mapsto \hat{p}$ would lead to identical corrected times and time allowances but would result in horrible hyperbolic *decreasing* functions that would not be easy to compare graphically. And, as long as it is common to all the boats that race, what the q ranges over doesn't really matter — corrected times are calculated by mapping back to the domain of elapsed times so the q range is hidden — the only requirement is that the q's be totally ordered and the *cap* function order preserving.

Performance curve scoring is always completely computerized, so in use it has been less constrained by the best practices we have explored here. Mapping observed course-average pace through the inverse to *cap*: $v_{\text{wind}} \mapsto \hat{p}$ yields an *imputed wind speed* which will order boats correctly with regards to the handicapping but which is otherwise completely opaque — reporting this instead of corrected time would be a mistake — yet this was typical for performance curve scoring when it was first introduced. ORC and ORR have corrected IMS practice in this regard — reporting an intelligible corrected time is valid whatever the style of handicapping (it's too bad ORC didn't stop here when it had got things right).

Some boats may have on-board computers that can calculate time allowances on the fly, but for everyone else precomputed tables of time allowances are essential. For simple time-on-distance or time-on-time handicapping this has usually been be left up to the competitor but for performance curve scoring the race organization should be involved to see that such tables are distributed in a timely manner — this has rarely occurred in practice. Such avoidable failures have set back the adoption of an otherwise promising handicapping technique.

Performance curve scoring isn't suitable for mental arithmetic but it can be approximated by time-on-time-and-distance handicapping making it not out reach for club racing. And with precomputed tables (and possibly graphs) of time allowances per boat, it might even be suitable for a club invitational regatta.

Chapter 8

A General Purpose Handicap

Time-on-distance and time-on-time are the only sensible single-factor styles of handicapping. And while time-on-time-and-distance subsumes them both it does so at the cost of greater complexity. In this chapter we will recapitulate some of the previous chapter but shall do so in the context of a *general purpose handicap* (abbreviated GPH and denoted by a variable g). In this restricted scope comparison between the single factor handicaps becomes easier.

Recall that a general purpose handicap g, being a pace that represents a boat's average performance, differs from a time-on-distance handicapping factor h or a time-on-time handicapping factor k only in that the same g can be used in either context: $h = g$ or $k = g$ interchangeably. And, while it may not be obvious at this point, all single-factor handicapping can be reformulated it terms of a general purpose handicap without changing corrected times in any way.

8.1 Applying General Purpose Handicaps to get Corrected Times

To determine how boats place in a race we rank them by how well each performs relative to its own standard pace, $p - g$ for time-on-distance and ${}^{p}\!/_{g}$ for time-on-time. Corrected paces are calculated

$$\check{\check{p}} = p + {}^\star g - g \quad \text{for time-on-distance}$$

$$\check{\check{p}} = p \times \frac{{}^\star g}{g} \quad \text{for time-on-time}$$

The *scratch handicap* ${}^\star g$, which is common for all boats, does not effect the handicapped finish order and is used to present corrected paces in a convenient form. Using the handicap of a boat in the race as ${}^\star g$ is best for a side-by-side comparison of time-on-distance to time-on-time. For the scratch boat the time-on-distance and time-on-time corrected pace will be the same as its observed pace $\check{\check{p}} = p$ (and likewise for corrected and elapsed times $\check{\check{t}} = t$). For any other boat the corrected pace formula $p \mapsto \check{\check{p}}$ maps observed pace for a boat to the pace it should have expected were it identical to the scratch boat.

Calculating corrected pace directly from the observed pace $p = {}^{t}\!/_{d}$ may be the most natural way to compare handicaps, but it is required by the *Racing Rules of Sailing* to multiply out by course distance to calculate corrected times from elapsed times

$$\begin{array}{rclcll} \check{\check{t}} & = & t + ({}^\star g - g)d & = & t + {}^\star gd - gd & \text{for time-on-distance} \\ \check{\check{t}} & = & t \times \frac{{}^\star g}{g} & = & t \times \frac{{}^\star gd}{gd} & \text{for time-on-time} \end{array}$$

Multiplying a general purpose handicap by course distance gives an interval of time we will call a *course specific handicap*.

8.2 Time Allowances for Time-on-Distance versus Time-on-Time

For time-on-distance the time allowance varies with respect to course length, and is therefore fixed throughout a given race. Whereas for time-on-distance the time allowance varies with respect to elapsed time. At some elapsed time a time-on-time allowance will overtake a time-on-distance allowance.

8.2.1 Time-on-Time Handicapping on the Race Course

Consider the *time allowance* Δt as the difference in time between a pair of boats that shall correct out the same and Δg as the corresponding difference in handicaps. Then these must satisfy the proportionality

$$\Delta t : \Delta g \times 1\,\mathrm{mi} \overset{\text{in proportion}}{=} t : g \times 1\,\mathrm{mi}$$

Stripping *per-mile* from the handicaps, the ratio of the time-on-time time allowance Δt to the difference in handicap Δg is equal in proportion to the ratio of elapsed time t to the handicap g. From this we can derive a simple formula

$$\frac{\Delta t}{\Delta g} = \frac{t}{g} \qquad \Longrightarrow \qquad \Delta t = \frac{\Delta g}{g} \times t$$

But it is easier to use the proportionality directly. With a general purpose handicap this is easily done even without a precomputed time allowance table

> Let's work an example — the other boat has a $790\,\mathrm{s/mi}$ handicap — you are the faster boat with a $765\,\mathrm{s/mi}$ handicap — then for every $765\,\mathrm{s} = 12\,\mathrm{min}\ 45\,\mathrm{s}$ of elapsed time you must gain $790\,\mathrm{s} - 765\,\mathrm{s} = 25\,\mathrm{s}$ on your competitor. If you finish with an elapsed time of $1\,\mathrm{h}\ 30\,\mathrm{min}$ which is approximately $7 \times 12\,\mathrm{min}\ 45\,\mathrm{s}$ then you need to win by approximately $7 \times 25\,\mathrm{s} = 2\,\mathrm{min}\,55\,\mathrm{s}$.
>
> Note that you can calculate time allowances for all the boats you are racing against by adding the Δg appropriate for each boat every $765\,\mathrm{s} = 12\,\mathrm{min}\ 45\,\mathrm{s}$ of your own elapsed time. The time interval is only dependent on your own handicap making it easy to track all your competitors simultaneously with only a little bit of preparation.

8.2.2 Time Allowances for Time-on-Distance versus Time-on-Time

Consider the time allowances for time-on-distance and time-on-time in comparison to each other. The time allowance for time-on-distance $\Delta t = \Delta gd$ (the difference in course specific handicaps) does not change throughout the race whereas time allowance for time-on-time increases proportionally throughout. As proportionalities in comparable terms this is expressed

$$\begin{aligned} \Delta t : \Delta gd &\overset{\text{in proportion}}{=} 1 : 1 && \text{for time-on-distance} \\ \Delta t : \Delta gd &\overset{\text{in proportion}}{=} t : gd && \text{for time-on-time} \end{aligned}$$

the ratio of the time allowance Δt to the difference in course specific handicap Δgd is one to one for time-on-distance and is equal in proportion to the ratio of elapsed time to course specific handicap for time-on-time. The ratio on the right can refer to either boat (for purposes of argument consider it your own boat). Whenever your elapsed time $t = gd$ then the time allowance is identical whether using time-on-distance or time-on-time handicapping. When $t < gd$ (i.e. your pace is faster than your standard pace) then the time allowance is less using time-on-time then it would be for time-on-distance handicapping. When $t > gd$ (i.e. your pace is slower than your standard pace) then the time allowance is greater using time-on-time then it would be for time-on-distance handicapping.

8.2.3 Which is Better?

From the presentation above, it would seem time-on-distance or time-on-time handicapping can be used interchangeably — after all, sometimes a boat will sail faster than its average pace and sometimes slower. The choice would seem to come down to which better models the actual relative performance of boats in varying conditions. The overwhelming consensus, supported by the velocity prediction programmes used for measurement handicaps, is that time-on-time handicapping can do the job better than time-on-distance.

There is a complication which makes the use of the same handicap for both time-on-distance and time-on-time difficult. For a boat to have a standard pace it really needs to sail in standard conditions — and average conditions clearly differ from place to place. It would seem that the best method for localizing such a system of handicaps is to honour the time-on-time model and scale all the standard paces by a common factor to reflect the standard conditions for the place. Note the ordering of corrected times using time-on-time handicaps are unaffected by such a transformation — time allowances for (the less representative) time-on-distance handicaps would be scaled appropriately for the average conditions. Localizing a system of handicaps by simply shifting by a common offset would preserve the ordering of corrected times using time-on-distance handicapping but is known to be less accurate than scaling.

Now as long as either time-on-distance or time-on-time handicapping is used consistently only the relative performance of boats is relevant to either the computation or application of handicaps — this can obscure the underlying dependence on average conditions when using both — and obscure the systemic weakness of time-on-distance handicaps that are not properly localized.

8.3 PHRF

8.3.1 The Relative Gauge

Within the regime of time-on-distance PHRF handicapping across North America the focus on relative performance (see also 10 Absolute versus Relative Performance) means that the expected differences in absolute paces have been largely ignored, with little or no attempt to reconstruct absolute paces until the recent shift to time-on-time handicapping. Traditional PHRF numbers, which were meant to be used only for time-on-distance handicapping, have been defined relative to a boat with a specified zero PHRF rating.

We add 557 s/mi to PHRF numbers to reconstruct a standard pace and general purpose handicap g. As long as you restrict your attention to time-on-distance handicapping the shifted and unshifted gauges behave identically. The 557 s/mi stated above is not part of the traditional definition of PHRF numbers but a modern reconstruction of the absolute pace implicit in the use of time-on-distance handicapping. In standard conditions a zero PHRF rated boat is expected to travel at an average pace of 557 s/mi.

8.3.2 Transitioning from Time-on-Distance to Time-on-Time

But there is a huge catch here. If ours is a light-air lake then the reconstructed standard paces via the offset $g = \text{PHRF} + 557\,\text{s/mi}$ will be faster than can be reasonably expected and resulting time allowances between boats using time-on-time handicapping will be greater than for time-on-distance more often than not. So the game really does change, though whether that is enough to alter the final ranking of finishes depends on how close the racing really is. The justification for this seems to be that the local time-on-distance handicaps are faulty, they show systemic bias inherent to using ratings from other stations which have greater winds and less difference in their expected paces.

Local clubs on the lake may accept this and use the $g = \text{PHRF} + 557\,\text{s/mi}$ conversion for subsequent time-on-time races. The counter argument is that the local PHRF station has already adapted their ratings over the years to properly account for local conditions using time-on-distance handicapping and therefore the $+557\,\text{s/mi}$ offset is too small.

Local clubs lack sufficient data to know the best value for this offset. The central PHRF authority complicates this by suggesting that race organizers determine the offset rather than the local handicapping station. Even worse, the central PHRF authority does not stress that this offset is a property of the venue, and not to changed by an RC to match the day; doing so would sacrifice all the conveniences of time-on-distance while maintaining all its weaknesses.

8.3.3 Over-correcting

Lake Ontario PHRF doesn't even try to reconstruct a realistic standard pace and uses an unrealistically small (and peculiarly precise) $+401.431\,\text{s/mi}$ offset to compensate for what they consider poor time-on-distance handicaps. While it is certainly easy to incorporate such adjustment factors into a corrected time formula — being no more than tweaking a single number — it does undercut the physical interpretation of such handicaps and lead to wildly different results using time-on-distance versus time-on-time handicapping.

8.3.4 Other Possible Reconstructions

Given the uncertainty in the $+557\,\text{s/mi}$ reconstruction it would perhaps have been better to have chosen a default offset more appropriate for mental arithmetic $+600\,\text{s/mi} = +10\,\text{min/mi}$. At a local level, it is better to explicitly correct for any historical bias in each boat's rating than to implicitly re-rate boats and silently ignore the systemic differences between time-on-time and time-on-distance handicapping.

8.3.5 Corrected Time Formulae in the PHRF Gauge

Using a zero PHRF rated boat as scratch the time-on-distance formula for corrected times takes on a particularly simple form with the balancing $\bigstar$ term eliminated

$$\check{\check{t}} \;=\; t - \text{PHRF} \times d$$

This choice of gauge ensured that corrected times were easy to calculate by hand using only positive numbers but is now annoying. It would be more convenient for competitors to have the standard paces published.

And a zero PHRF rated boat is rarely a good choice for a scratch boat. Letting ${}^{\bigstar}\text{PHRF}$ be the PHRF number for a scratch boat the corrected time formulae need to be tweaked to accommodate the shifted gauge

$$\begin{aligned} \check{\check{t}} &= t + ({}^{\bigstar}\text{PHRF} - \text{PHRF}) \times d && \text{for time-on-distance} \\ \check{\check{t}} &= t \times \frac{{}^{\bigstar}\text{PHRF}+557\,\text{s/mi}}{\text{PHRF}+557\,\text{s/mi}} && \text{for time-on-time} \end{aligned}$$

These are the corrected time formulae after having substituted for $g = \text{PHRF} + 557\,\text{s/mi}$ and ${}^{\bigstar}g = {}^{\bigstar}\text{PHRF} + 557\,\text{s/mi}$ and then having simplified. The transformed formulae are not much more complicated than those that are defined in terms of a general purpose handicap but they are inelegant and their physical significance is obscured.

8.3.6 Dead Weight

PHRF numbers carry a lot of historical dead weight. For time-on-time handicapping we would be better served to have ratings with the $+557\,\mathrm{s/mi}$ shift already incorporated. And for time-on-distance handicapping these numbers should be scaled to better reflect expected paces.

8.3.7 More Dead Weight

For time-on-time handicapping, another historical albatross is the publication and use of *time correction factors*

$$\mathrm{TCF} = \frac{{}^\star\mathrm{PHRF}+557\,\mathrm{s/mi}}{\mathrm{PHRF}+557\,\mathrm{s/mi}}$$

These are even more awkward to deal with than the underlying PHRF numbers as they must be inverted before being used in a proportionality to determine time allowances. Now the inverse itself

$$\frac{\mathrm{PHRF}+557\,\mathrm{s/mi}}{{}^\star\mathrm{PHRF}+557\,\mathrm{s/mi}}$$

is convenient for working with time allowances and, when expressed as a decimal fraction or a percentage, may be considered a useful analogue to a Portsmouth handicap. Whichever form is used, the use of a single scratch handicap for all divisions will result in obscure corrected times for most boats.

8.3.8 Summary

In summary, these are the corrected time formulae in common use after reconstructing a meaningful standard pace and general purpose handicap $g = \mathrm{PHRF} + 557\,\mathrm{s/mi}$

In conventional form for time-on-distance with a scratch ${}^\star\mathrm{PHRF} = 0\,\mathrm{s/mi}$

$$\begin{array}{lcccc}
\text{time-on-distance} & \check{\check{t}} = & t - (g - 557\,\mathrm{s/mi}) \times d & = & t - \mathrm{PHRF} \times d \\
\text{time-on-time} & \check{\check{t}} = & t \times \frac{557\,\mathrm{s/mi}}{g} & = & t \times \frac{557\,\mathrm{s/mi}}{\mathrm{PHRF}+557\,\mathrm{s/mi}}
\end{array}$$

In conventional form for time-on-time with a scratch ${}^\star\mathrm{PHRF} = 93\,\mathrm{s/mi}$

$$\begin{array}{lcccc}
\text{time-on-distance} & \check{\check{t}} = & t - (g - 650\,\mathrm{s/mi}) \times d & = & t - (\mathrm{PHRF} - 93\,\mathrm{s/mi}) \times d \\
\text{time-on-time} & \check{\check{t}} = & t \times \frac{650\,\mathrm{s/mi}}{g} & = & t \times \frac{650\,\mathrm{s/mi}}{\mathrm{PHRF}+557\,\mathrm{s/mi}}
\end{array}$$

In general form with an arbitrary scratch ${}^\star g = {}^\star\mathrm{PHRF} + 557\,\mathrm{s/mi}$

$$\begin{array}{lcccc}
\text{time-on-distance} & \check{\check{t}} = & t - (g - {}^\star g) \times d & = & t - (\mathrm{PHRF} - {}^\star\mathrm{PHRF}) \times d \\
\text{time-on-time} & \check{\check{t}} = & t \times \frac{{}^\star g}{g} & = & t \times \frac{{}^\star\mathrm{PHRF}+557\,\mathrm{s/mi}}{\mathrm{PHRF}+557\,\mathrm{s/mi}}
\end{array}$$

For time-on-distance the time allowances Δt for boats to correct out to the same are

$$\Delta t = \Delta g \times d = \Delta\mathrm{PHRF} \times d$$

And for time-on-time the time allowances Δt are determined by the proportionalities

$$\begin{aligned}
\Delta t : \Delta g \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t : g \times 1\,\mathrm{mi} \\
\Delta t : \Delta\mathrm{PHRF} \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t : \mathrm{PHRF} \times 1\,\mathrm{mi} + 557\,\mathrm{s}
\end{aligned}$$

i.e. stripping *per-mile* from the handicaps gives a distance independent proportionality. Or maintaining the units and expressing these proportionalities in the form of an equation we can write

$$\frac{\Delta t}{\Delta g} = \frac{t}{g} \qquad\qquad \frac{\Delta t}{\Delta\mathrm{PHRF}} = \frac{t}{\mathrm{PHRF} + 557\,\mathrm{s/mi}}$$

Chapter 9

Interpreting Intervals of Corrected Time

The definition of corrected time comes with a built-in metaphor, and the flexibility to recalculate a race with your own boat as scratch relieves you of the burden of having to think about it too much. And time-on-distance handicapping is so simple it hardly needs mentioning. But there is a another metaphor, a trick to interpreting intervals of corrected time with time-on-time and time-on-time-and-distance handicapping that makes reading results more accessible.

9.1 Intervals of Corrected Time versus Intervals of Elapsed Time

In this chapter we will use the $\Delta\check{\check{t}}$ notation (with two checks on top) to refer to any interval of corrected time and Δt to refer to a corresponding interval of elapsed time (not a time allowance). For a scratch handicap ${}^\star h$, ${}^\star k$ or $\left[{}^\star k \quad {}^\star h\right]$ and a boat with handicap h, k or $\left[k \quad h\right]$, any interval of corrected time $\Delta\check{\check{t}}$ is related to the corresponding interval of elapsed time Δt only by the time-coefficient of the handicaps. So for time-on-distance handicapping the intervals are always identical. For time-on-time and time-on-time-and-distance

$$\frac{\Delta\check{\check{t}}}{{}^\star k} = \frac{\Delta t}{k}$$

Or in terms of a proportion

$$\Delta\check{\check{t}} : \Delta t \quad \overset{\text{in proportion}}{=} \quad {}^\star k : k$$

9.1.1 Example Ratios of the Time Coefficients of Handicaps

Here are some good approximate ratios (the best with terms less than 60) of the scratch time coefficient ${}^\star k$ to each boat's time coefficient k using the example time-on-time and time-on-time-and-distance handicaps and using Hurricane as scratch

Examples for Time-on-Time & Time-on-Time-and-Distance

Boat	k	$^{\bigstar}k{:}k$	$\begin{bmatrix}k & h\end{bmatrix}$	$^{\bigstar}k{:}k$	Make
Hurricane	729	★	[672 723]	★	Buddy 24
Winged Elephant	810	9:10	[666 807]	60:59	Frequency 24
Mechanical Drone	834	7:8	[813 840]	43:52	See in Sea 30
Shindig	861	11:13	[858 858]	47:60	Raider 28
Rhumb Punch	876	5:6	[954 876]	31:44	Chimera 33

By happenstance, the time-on-time ratios are much nicer. Using either time-on-time or time-on-time-and-distance would make no difference to the following presentation, so we will use the nicer ratios.

9.2 An Example Time-on-Time Race with *Hurricane* as Scratch

Boat	Handicap	Elapsed Time	Corrected Time	$\Delta\check{\check{t}}$
Shindig	861	1:29:59	1:16:11.3	−61.7
Rhumb Punch	876	1:31:45	1:16:21.2	−51.8
Hurricane	729	1:17:13	1:17:13.0	★
Mechanical Drone	834	1:29:29	1:18:13.0	+60.0
Winged Elephant	810	1:27:01	1:18:18.9	+65.9

The columns show handicaps k, elapsed times t, corrected times $\check{\check{t}}$ (rounded to the nearest tenth of a second) and how much later than the scratch boat Hurricane that each boat finished in corrected time $\Delta\check{\check{t}}$ (rounded likewise).

9.2.1 From The Scratch Boat *Hurricane's* Point of View

The scratch boat itself is in an enviable position that its corrected time and elapsed times are the same and the corrected times of competitors' boats are directly comparable to its own elapsed time. Hurricane can see, directly, how well it would have placed had it finished 62 s sooner, between 61 s and 52 s sooner, greater than 1 min later and so on (and had all the other boats finished as before). Corrected times are displayed rounded to the nearest tenth of a second so these reported times lack the precision required to break the potential tie with *Mechanical Drone* had Hurricane finished exactly 1 min later.

9.2.2 From *Rhumb Punch's* Point of View Without Recalculating Results

$$729\,\mathrm{s/mi} : 876\,\mathrm{s/mi} \overset{\text{in proportion}}{\approx} 5:6 \overset{\text{in proportion}}{\approx} +52\,\mathrm{s} : +62\,\mathrm{s}$$

Every 6 s of elapsed time for Rhumb Punch evaluates to approximately 5 s of corrected time. Rhumb Punch looks at race results where it is 52 s ahead of Hurricane in corrected time. Had Rhumb Punch finished 63 s later it would have dropped a place.

$$5:6 \overset{\text{in proportion}}{=} 10\,\mathrm{s} : 12\,\mathrm{s}$$

Had Rhumb Punch finished 12 s sooner it would catch up and pass *Shindig*.

9.2.3 From *Mechanical Drone's* Point of View Without Recalculating Results

$$729\,\mathrm{s/mi} : 834\,\mathrm{s/mi} \overset{\text{in proportion}}{\approx} 7 : 8 \overset{\text{in proportion}}{=} 112\,\mathrm{s} : 128\,\mathrm{s}$$

Every 7 s of elapsed time for Mechanical Drone evaluates to approximately 8 s of corrected time. Mechanical Drone looks at race results where it is 111.8 s behind *Rhumb Punch* in corrected time. Had Mechanical Drone finished 128 s sooner it would catch up to Rhumb Punch.

9.3 A Cheap Metaphor

Think of a second of corrected time as the international currency of exchange US$1.

Boat	Handicap	Conversions		Common Currency
Shindig	861	US$0.85 = Sh$1	US$1 = Sh$1.18	−US$61.7
Rhumb Punch	876	US$0.83 = RP$1	US$1 = RP$1.20	−US$51.8
(US) Hurricane	729	★		
Mechanical Drone	834	US$0.87 = MD$1	US$1 = MD$1.14	+US$60.0
Winged Elephant	810	US$0.90 = WE$1	US$1 = WE$1.11	+US$65.9

Winged Elephant's currency doesn't quite measure up at a fixed exchange rate of WE$1 = US$0.90 or US$1 = WE$1.11. Winged Elephant spent an additional US$65.90 in the currency of exchange more than Hurricane. From Winged Elephant's point of view, had it spent US$66 = WE$73.25 less it would have beaten Hurricane. Abandoning the metaphor for a moment, 73.25 s of elapsed time for Winged Elephants is comparable to 66 s of corrected time (or 66 s of elapsed time for the scratch boat Hurricane). From *Mechanical Drone's* point of view had it spent US$60 = MD$68.65 less it would have caught up to Hurricane or US$6 = MD$6.75 more and it would have lost to Winged Elephant.

The message to take is that elapsed times are not directly comparable across boats (unless they have same handicap). Like currency, seconds of elapsed time need to be tagged with the context in which they can be spent without conversion. Converting to a common currency of corrected time (which is elapsed time for the scratch boat) allows for comparison across many boats.

9.3.1 Small Change

Even though converting between elapsed times for one boat to elapsed times for another boat is as is conceptually simple as converting currencies, for boats other than the scratch boat such conversions quickly become tiresome. For classes with a small range of handicaps using a boat within the class as scratch would allow competitors to approximate small intervals of elapsed time with the same interval of corrected time. If the difference between handicaps is small this works well.

9.4 The Same Race with Each Boat as Scratch

It would be best for any boat to be able to see the results with itself as the scratch boat. Let's reexamine our race with each boat as scratch, in turn. To make them all fit on the page, we wont show the corrected times $\check{\check{t}}$, just the differences $\Delta\check{\check{t}}$

Boat	Handicap	Elapsed Time	Differences in Corrected Time				
Shindig	861	1:29:59	★	−11.9	−61.7	−139.3	−141.8
Rhumb Punch	876	1:31:45	+11.7	★	−51.8	−127.9	−130.8
Hurricane	729	1:17:13	+72.9	+62.2	★	−68.7	−73.2
Mechanical Drone	834	1:29:29	+143.8	+134.4	+60.0	★	−6.5
Winged Elephant	810	1:27:01	+150.7	+141.4	+65.9	+6.7	★

Let's reexamine our race with *Winged Elephant* as the scratch boat. The ordering of finishes as indicated by the final column is unchanged from when Hurricane or any other boat were scratch. But for this competitor conversion between elapsed and corrected times is no longer necessary as times in the final column can be interpreted as intervals of elapsed time for Winged Elephant.

9.4.1 On the Web

It is rarely practical on paper to show an added column for each boat, but showing a whole new table for each boat but is easily achieved on the web. Clicking on a table row would (indeed should) recalculate corrected times and differences using the selected boat as scratch.

Chapter 10

Absolute versus Relative Performance

10.1 Relative Gauge Handicaps

10.1.1 Using the Scratch Boat as a Standard of Performance Prediction

Traditionally, handicaps were an expression of relative performance applicable to very simple corrected time formulae. For each boat, using its absolute performance prediction h, k or $\begin{bmatrix} k & h \end{bmatrix}$, we can derive an alternative handicap $h_\star$, $k_\star$ or $\begin{bmatrix} k_\star & h_\star \end{bmatrix}$ that gauges performance relative to the standard boat $\bigstar$ and from this calculate corrected time with respect to the standard boat as scratch

$$\begin{array}{llll} \check{\check{t}} = t - h_\star d & \text{using the derived} & h_\star = h - {}^\bigstar h & \text{for time-on-distance} \\ \check{\check{t}} = \dfrac{t}{k_\star} & \text{using the derived} & k_\star = \frac{k}{{}^\bigstar k} & \text{for time-on-time} \\ \check{\check{t}} = \dfrac{t - h_\star d}{k_\star} & \text{using the derived} & k_\star = \frac{k}{{}^\bigstar k},\quad h_\star = h - \frac{k}{{}^\bigstar k}\,{}^\bigstar h & \text{for time-on-time-and-distance} \end{array}$$

Note that this gives an alternative formula for corrected time but not an alternative definition.

10.1.2 Disseminating Relative Performance Predictions as Handicaps

For the standard boat $\bigstar$ itself we will have $k_\star = 1.000$ ($1000\,{}^{\mathrm{s}}/_{\mathrm{ks}}$) and $h_\star = 0\,{}^{\mathrm{s}}/_{\mathrm{mi}}$. All other time coefficients $k_\star$ will be unitless fractions near to one (about one thousand seconds per kilosecond). The distance coefficients $h_\star$ will be signed offsets in units of seconds per nautical mile. As a way to calculate intermediate terms for a corrected time formula, this is entirely equivalent to what we have already seen. However, should each boat be given a rating certificate with the time and distance coefficients $k_\star$ and $h_\star$ (which we will call the *relative gauge*) instead of the k and h (which we now call *absolute gauge* handicaps):

- the handicaps become less intuitive
- the corrected time formulae become simpler (?)
- the scratch boat becomes fixed by the choice embedded in the relative gauge handicaps (?)
- performance line plots and critical pace plots relative to the standard boat become the same
- the distance coefficients of time-on-time-and-distance handicaps become very hard to interpret
- the critical proportions for time allowances still hold but the units differ
- it becomes harder to start a table of time allowances for time-on-time-and-distance handicaps

10.1.3 The Absolute Gauge and Relative Gauge Handicaps Have Equal Footing

That is to say, all the defining equations, formulae and proportionalities of the previous sections hold as much for these relative gauge handicaps as they did for the absolute (gauge) performance predictions we used to introduce them. The units for time coefficients of handicaps and the resultant $\check{q}$ and q will differ without deleterious effect. And we need write relative gauge handicaps with a $\star$ subscript only so long as it is convenient to do so — for time-on-time and time-on-time-and-distance they are already differentiated by their units.

10.1.4 The Simplified Corrected Time Formulae and Hand Computation Bias

The simplified corrected time formula where the corrected time $\check{\check{t}}$ is determined directly from elapsed time t does not supersede the general formula or the defining equations for corrected time. Rather this is a special case — the standard boat will have a ${}^{\bigstar}k_{\star} = 1.000$ ($1000\,\mathrm{s/ks}$) and ${}^{\bigstar}h_{\star} = 0\,\mathrm{s/mi}$ handicap — should we use this as the scratch boat in the general corrected time formulae it is easy to see that the corrected time $\check{\check{t}}$ would then be identical to $\check{u}$.

There is an opportunity to use the simplified corrected time formula but it is not strictly necessary or even desirable to do so — if we don't use this preferred scratch we can still use the general formulation. Indeed, we should always use a scratch boat suitable for each division and for results on the web we should always allow competitors to select their own boat as scratch. The supposed advantage of a simplified corrected time formula is moot — we will never want to take advantage of it.

Historically, rating and handicapping rules went out of their way to simplify the mechanisms of calculating corrected times to make them suitable for computation by hand. Combined with the fragility of determining ties when using a rounding rule, a practical necessity for hand computation, being locked into a choice of scratch boat for the sake of a simplified corrected time formula seemed reasonable. Today, there is no such justification.

Indeed, the only reason to use relative gauge handicaps at all is their historical relevance. New measurement rules would do best to avoid them altogether. They offer no advantage over the more intuitive absolute gauge handicaps.

10.1.5 Three Similar Annotations: the Circle $\circ$, Big Star $\bigstar$ and Little Star $\star$

There is also a notational subtlety we need deal with here. The general formulation of the corrected time formula will identify one boat $\bigstar$ as scratch. This boat is chosen by the race organizers or the race committee or perhaps each competitor after the fact. The standard boat $\star$ is best thought of as a gauge by which published handicaps are measured and has no relevance to the determination of corrected times — i.e. the scratch boat should not coincide with the standard boat as chosen by the handicapping authority. We can use the big star $\bigstar$ and small star $\star$ when we must differentiate between these chosen boats. And one final subtlety: we may not have the luxury of being to choose our own boat as scratch in published results; when we need to pick out our own boat $\circ$ yet, in the same context, differentiate it from both the scratch boat and the standard boat we can use a circle $\circ$. Contrarily, within this book, we are using the circle in several other contexts for completely different purposes.

10.1.6 Critical Proportions for Time Allowances and Tables

To construct tables of time allowances and easily interpret the critical proportions we need to be able to interpret the $k_\star$ and $\Delta k_\star$ as intervals of time. If these are given as decimal numbers to three digits of precision, multiplying them by 1 ks = 1000 s (one kilosecond) will give a direct and reasonably straightforward interpretation to the unitless $k_\star$ — writing the time coefficients in units of $\mathrm{s/ks}$ (seconds per kilosecond) supports this usage — the one hitch is that 1000 s = 16 min 40 s making the conversion to minutes and seconds a bit error-prone.

$$\begin{aligned} &1000\,\mathrm{s} && = 16\,\mathrm{min}\,40\,\mathrm{s} \\ &1000\,\mathrm{s} \times 3/5 = 600\,\mathrm{s} && = 10\,\mathrm{min} \\ &1000\,\mathrm{s} \times 1/5 = 200\,\mathrm{s} && = 3\,\mathrm{min}\,20\,\mathrm{s} \end{aligned}$$

Having handicaps rounded to the nearest five seconds per kilosecond for a time coefficent or the nearest five seconds per mile for a distance coefficent will ensure that the numbers in the body of a time allowance tables advancing by fifths will remain whole numbers; and every third row of such a table will advance with a ten minute cadence (exactly 10 min when the time coefficient $k = 1000\,\mathrm{s/ks}$).

10.1.7 Performance Lines Relative to the Standard Boat

For course-average and corrected paces

on Distance

$$\check{\check{p}} = p - h_\star$$

on Time

$$\check{\check{p}} = \frac{p}{k_\star}$$

on Time and Distance

$$\check{\check{p}} = \frac{p - h_\star}{k_\star}$$

For performance lines

$$\begin{aligned} \hat{p} &= h_\star + q && \text{for time-on-distance} \\ \hat{p} &= k_\star q && \text{for time-on-time} \\ \hat{p} &= h_\star + k_\star q && \text{for time-on-time-and-distance} \end{aligned}$$

In this context the variable q is the nominal course-average pace expected for a $\star$ standard boat.

In a graph of performance lines the relative gauge distance coefficients $h_\star$ would be the intercepts on $q = 0$ axis (i.e infinite speed for the standard boat), off to the left of the plot and well outside the range in which the handicapping lines are applicable. This can make the $h_\star$ hard to interpret, particularly when the differences in time coefficient $\Delta k_\star$ are large. When the $\Delta k_\star = 0$, as it for time-on-distance handicapping, there is no difficulty in interpreting the $h_\star$.

10.1.8 Time-on-Time-and-Distance Performance Handicaps Recapitulated

Here are the example time-on-time-and-distance performance handicaps in the gauge relative to the standard boat $\star$ *Shindig* and rounded a little differently than before for the benefit of the following examples. The units in the table will be $[\mathrm{s/ks} \quad \mathrm{s/mi}]$.

Boat	Handicap	Handicap Differences			Make
Hurricane	[785 +50]	★	[−165 +25]	[−215 +50]	Buddy 24
Winged Elephant	[775 +140]	[−10 +90]	[−175 +115]	[−225 +140]	Frequency 24
Mechanical Drone	[950 +25]	[+165 −25]	★	[−50 +25]	See in Sea 30
⋆ Shindig	[1000 0]	[+215 −50]	[+50 −25]	★	Raider 28
Professor	[1125 −95]	[+340 −145]	[+175 −120]	[+125 −95]	Stone 22
Rhumb Punch	[1115 −75]	[+330 −125]	[+165 −100]	[+115 −75]	Chimera 33

(see subsection 7.4.2 for the same in an absolute gauge)

The difference of relative gauge distance coefficents $\Delta h_\star$ can be difficult to interpret. A $\Delta h_\star$ of zero indicates a time-on-time relationship between boats. A $\Delta h_\star$ of the same sign as $\Delta k_\star$ indicates the performance between the boats diverges as the wind lessens but at a slower rate than you would expect for a time-on-time relationship. A $\Delta h_\star$ of the opposite sign to $\Delta k_\star$ can indicate one of three possibilites: that the performance between the boats diverges as the wind lessens but at a faster rate than you would expect for a time-on-time relationship; that their performance actually intersects somewhere in the range of accessible course-average paces; or that performance converges as the wind lightens but not sufficiently to intersect at a reasonable pace — which of these alternatives holds true isn't immediately obvious, but as the $\Delta h_\star$ gets larger in magnitude with respect to the $\Delta k_\star$ the further to the right of the plot the point of intersection will occur. In particular, it may not be immediately obvious which of a pair of boats will perform better in normal conditions based upon their $\begin{bmatrix} k_\star & h_\star \end{bmatrix}$ handicaps.

For boats that race in the same division of a handicapped class you would expect their $\Delta h_\star$ to have an opposite sign to $\Delta k_\star$; these boats have already been grouped together based on their average performance so what differences remain would show up at the extremes.

10.1.9 Time Allowances Table for Time-on-Time-and-Distance or Time-on-Time

For the time-on-time-and-distance tables, there will no longer be a centre row to the tables, instead they will need to build up from the bottom as does the time-on-time table. But making a table of time-on-time-and-distance allowances using the proportionality is still straightforward. For every $k_\star \times 1\,\text{ks}$ in excess of $h_\star d$ that you spend on the course, the base allowance of $\Delta h_\star d$ goes up by $\Delta k_\star \times 1\,\text{ks}$. With $\tau = 1\,\text{ks}$ we have tables for time-on-time-and-distance and time-on-time, respectively

archetype for time-on-time-and-distance

	Elapsed Time	$\cdots$	Time Allowance	$\cdots$
	$h_\star d$		$\Delta h_\star d$	
τ	$h_\star d + k_\star \tau$		$\Delta h_\star d + \Delta k_\star \tau$	
2τ	$h_\star d + 2k_\star \tau$		$\Delta h_\star d + 2\Delta k_\star \tau$	
3τ	$h_\star d + 3k_\star \tau$		$\Delta h_\star d + 3\Delta k_\star \tau$	
4τ	$h_\star d + 4k_\star \tau$		$\Delta h_\star d + 4\Delta k_\star \tau$	
5τ	$h_\star d + 5k_\star \tau$		$\Delta h_\star d + 5\Delta k_\star \tau$	
$\vdots$	$\vdots$		$\vdots$	
	table only holds for course length d			

archetype for time-on-time

	Elapsed Time	$\cdots$	Time Allowance	$\cdots$
	$k_\star \tau$		$\Delta k_\star \tau$	
τ	$2k_\star \tau$		$2\Delta k_\star \tau$	
2τ	$3k_\star \tau$		$3\Delta k_\star \tau$	
3τ	$4k_\star \tau$		$4\Delta k_\star \tau$	
4τ	$5k_\star \tau$		$5\Delta k_\star \tau$	
5τ	$6k_\star \tau$		$6\Delta k_\star \tau$	
6τ	$7k_\star \tau$		$7\Delta k_\star \tau$	
7τ	$8k_\star \tau$		$8\Delta k_\star \tau$	
$\vdots$	$\vdots$		$\vdots$	

Nonsensical negative elapsed times could easily pop up in the first few rows of the time-on-time-and-distance table — those rows need to be discarded.

10.1.10 Examples of Time-on-Time-and-Distance Tables of Allowances

The calculation of corrected times is simplified when the standard boat is selected as scratch. Likewise, a table of time allowances for use by standard boat itself will be keyed as if to a stopwatch, at conveniently simple-to-express elapsed times. Note that three fifths of a kilosecond is 10 min and we have rounded our handicaps factors to the nearest multiple of five (in the appropriate units s/ks or s/mi) to support tables of time allowance with rows advancing by fifths using $\tau = 1/5\,\text{ks} = 200\,\text{s} = 3\,\text{min}\,20\,\text{s}$. For the standard boat there will be exactly three rows of the table every ten minutes.

Example of Time Allowances From the Perspective of the Standard Boat Your boat *Shindig* is the standard for the relative gauge with a handicap of $\begin{bmatrix} 1000\,\text{s/ks} & 0\,\text{s/mi} \end{bmatrix}$. The course is 4 mi.

For the standard boat the base time is always zero. The time allowance for each competitor will start with the base allowance $\Delta h_\star \times 4\,\text{mi}$ at zero elapsed increasing by $\Delta k_\star \times {}^1\!/_5\,\text{ks}$ for every $3\,\text{min}\,20\,\text{s}$ of elapsed time where $3\,\text{min}\,20\,\text{s} = 200\,\text{s} = {}^1\!/_5\,\text{ks}$ is just underlying interval τ as befits the standard boat.

Second Perspective for Time-on-Time-and-Distance Time Allowances Your boat *Mechanical Drone* has a handicap of $\left[950\,{}^{\text{s}}\!/_{\text{ks}} \quad +25\,{}^{\text{s}}\!/_{\text{mi}}\right]$. Note that 950 s is 15 min 50 s. The course is 4 mi. $+25\,{}^{\text{s}}\!/_{\text{mi}} \times 4\,\text{mi} = 1\,\text{min}\,40\,\text{s}$. In increments of a fifth $15\,\text{min}\,50\,\text{s} \times {}^1\!/_5 = 3\,\text{min}\,10\,\text{s}$. The time allowance for each competitor will be $\Delta h_\star \times 4\,\text{mi}$ at 1 min 40 s of elapsed time (0:01:40) increasing by $\Delta k_\star \times {}^1\!/_5\,\text{ks}$ every 3 min 10 s.

Example Tables of Time Allowances for Time-on-Time-and-Distance on a 4 mi Course

★ 4	Shindig [1000 0]	Mech. D. [−50 +25]	Professor [+125 −95]	Rhumb P. [+115 −75]	Winged E. [−225 +140]	Hurricane [−215 +50]
2.4	0:40:00	−0:20	−1:20	−0:24	+0:20	−5:16
2.6	0:43:20	−0:30	−0:55	−0:01	−0:25	−5:59
2.8	0:46:40	−0:40	−0:30	+0:22	−1:10	−6:42
3	0:50:00	−0:50	−0:05	+0:45	−1:55	−7:25
3.2	0:53:20	−1:00	+0:20	+1:08	−2:40	−8:08
3.4	0:56:40	−1:10	+0:45	+1:31	−3:25	−8:51
3.6	1:00:00	−1:20	+1:10	+1:54	−4:10	−9:34
3.8	1:03:20	−1:30	+1:35	+2:17	−4:55	−10:17
4	1:06:40	−1:40	+2:00	+2:40	−5:40	−11:00
4.2	1:10:00	−1:50	+2:25	+3:03	−6:25	−11:43
4.4	1:13:20	−2:00	+2:50	+3:26	−7:10	−12:26
4.6	1:16:40	−2:10	+3:15	+3:49	−7:55	−13:09
4.8	1:20:00	−2:20	+3:40	+4:12	−8:40	−13:52

★ 4	Mech. D. [950 +25]	Shindig [+50 −25]	Professor [+175 −120]	Rhumb P. [+165 −100]	Winged E. [−175 +115]	Hurricane [−165 +25]
2.4	0:39:40	+0:20	−1:00	−0:04	+0:40	−4:56
2.6	0:42:50	+0:30	−0:25	+0:29	+0:05	−5:29
2.8	0:46:00	+0:40	+0:10	+1:02	−0:30	−6:02
3	0:49:10	+0:50	+0:45	+1:35	−1:05	−6:35
3.2	0:52:20	+1:00	+1:20	+2:08	−1:40	−7:08
3.4	0:55:30	+1:10	+1:55	+2:41	−2:15	−7:41
3.6	0:58:40	+1:20	+2:30	+3:14	−2:50	−8:14
3.8	1:01:50	+1:30	+3:05	+3:47	−3:25	−8:47
4	1:05:00	+1:40	+3:40	+4:20	−4:00	−9:20
4.2	1:08:10	+1:50	+4:15	+4:53	−4:35	−9:53
4.4	1:11:20	+2:00	+4:50	+5:26	−5:10	−10:26
4.6	1:14:30	+2:10	+5:25	+5:59	−5:45	−10:59
4.8	1:17:40	+2:20	+6:00	+6:32	−6:20	−11:32

(see 7.4.5 for the same in an absolute gauge)

10.2 Conversions Between the Absolute and Relative Gauges

The mapping from an absolute to a relative gauge via a standard boat $h \mapsto h_\star$, $h \mapsto k_\star$ or $\left[k \quad h\right] \mapsto \left[k_\star \quad h_\star\right]$ can be easily inverted. We make use of a single absolute performance prediction for the standard boat ★

$$
\begin{array}{rcll}
h_\star = h - {}^\bigstar h & \iff & h = h_\star + {}^\bigstar h & \text{for time-on-distance} \\
k_\star = \dfrac{k}{{}^\bigstar k} & \iff & k = k_\star\, {}^\bigstar k & \text{for time-on-time} \\
k_\star = \dfrac{k}{{}^\bigstar k},\ h_\star = h - \dfrac{k}{{}^\bigstar k}\, {}^\bigstar h & \iff & k = k_\star\, {}^\bigstar k,\ h = h_\star + k_\star\, {}^\bigstar h & \text{for time-on-time-and-distance}
\end{array}
$$

Note that these conversions always map the gauge for the entire collection of handicaps (i.e. at least all the boats that have been issued certificates by the handicapping authority). While it is good practice to choose a different scratch boat for each division of a race, applying a gauge conversion to only one division of boats would be perverse.

10.2.1 Gauge Conversions Do Not Effect Corrected Times Whatsoever

Corrected times, defined with respect to a scratch boat, are unaffected by a gauge conversion.

Remember that a corrected time is a performance prediction of how a boat should have finished, given its elapsed time, were it the same as the scratch boat. Gauge conversions can be thought of as a unit conversion with which the performance prediction inherent to a handicap is expressed but when comparing one boat to another all those units cancel out of the formulation. It is unsurprising that, when handicapping is effectively unchanged, the only way to achieve different corrected times is to choose a different boat as scratch.

10.2.2 Each Absolute Gauge and Each Relative Gauge Has Equal Footing

Each different choice for the standard boat yields a different collection of handicaps all giving the same results. We can combine the conversions between the absolute and relative gauges to map any relative gauge to any other relative gauge. Not as obviously, we could also map any absolute gauge to any other absolute gauge. Take, for example, a venue with consistently light air. Knowing the handicap performance potential for a particular boat $\bigstar$ in general conditions $\begin{bmatrix} {}^{\bigstar}k & {}^{\bigstar}h \end{bmatrix}$ and the observed performance for that boat at the light air venue $\begin{bmatrix} {}^{\bigstar}k_{\circ} & {}^{\bigstar}h_{\circ} \end{bmatrix}$ we could map to the relative gauge with the $\begin{bmatrix} {}^{\bigstar}k & {}^{\bigstar}h \end{bmatrix}$ and then back to the absolute gauge with the $\begin{bmatrix} {}^{\bigstar}k_{\circ} & {}^{\bigstar}h_{\circ} \end{bmatrix}$ to localize the gauge for the entire collection of handicaps. This localization would not effect race results in any way but would make the handicaps more intuitive for the venue and keep created time allowance tables well centred around expected elapsed times.

Note that specifying a handicap for a single boat specifies the gauge for all boats. Most rating or handicapping rules specify a single gauge to be used by all boats by pinning the handicap of a *standard* boat. For new rating rules, whether that published gauge is for absolute or relative performance seems to be a matter of taste. We would argue that absolute gauge handicaps are better. Historically the published gauge was always relative and, for performance rules, this is still the case. Perversely, ORC and ORR publish time-on-distance handicaps in an absolute gauge with time-on-time variants in a relative gauge.

10.2.3 The General Purpose Handicap: A Well-Localized Absolute Gauge

Also note that, for single-factor handicapping, a well-localized absolute gauge time-on-time handicap can be used interchangeably with a well-localized absolute gauge time-on-distance handicap or with the distance coefficient of a well-localized absolute gauge time-on-time-and-distance handicap for either purpose. This is sometimes referred to as a *general purpose handicap* (GPH). On the other hand, the three separate styles of relative gauge handicaps each live in their own separate universe without obvious connection to one another.

10.2.4 Gauge Transformations

The mapping from one absolute gauge to another absolute gauge is called a *gauge transformation*. Likewise, the mapping from one relative gauge to another relative gauge is also called a *gauge transformation*. We should stress that specifying the transformation of a handicap for a single boat determines how it will transform the handicaps for all boats. And specifying how a gauge conversion will convert the handicap for a single boat determines how it will convert the handicaps for all boats.

Corrected times, defined with respect to a scratch boat, remain unaffected by any combination of gauge transformations and gauge conversions. This is the defining characteristic of a gauge transformation or conversion.

We have described a gauge transformation in terms of a pair of conversions back and forth, but there are better ways to describe a gauge transformation as a single operation. For any two gauges there exists a single gauge transformation or conversion that takes one to the other. Gauge transformations for time-on-distance and time-on-time are so straightforward that they really don't need elaboration — time-on-time-and-distance introduces complications we aren't yet ready to address.

10.2.5 Gauge Conversions and Units

For time-on-time and time-on-time-and-distance the conversion between the absolute and relative gauges also changes the units of a handicap's time coefficient, but for time-on-distance handicaps there is no such unit conversion making the difference between a gauge conversion and a gauge transformation one of degree, not kind.

Also note that there is no requirement that the units of the time coefficient of a handicap have any meaning whatsoever — in the formulae for corrected time all time coefficient units cancel out. We find it helpful to have a consistent interpretation for time coefficients in each of the regimes of absolute and relative performance and to keep the concepts of gauge conversion and gauge transformation separate.

10.2.6 Gauges of Preserved Dimensionality vs. Flattened Dimensionality

For all handicapping in a relative gauge but also for time-on-distance handicapping in an absolute gauge the *chk* function maps units of pace to units of pace. For time-on-time and time-on-time-and-distance handicapping in an absolute gauge the *chk* function maps units of pace to unitless numbers. The former we call *gauges of preserved dimensionality* and the latter *gauges of flattened dimensionality*.

10.2.7 Units in the Parametrization of Gauge Transformations

We can parametrize a gauge transformation using a unitless positive parameter e and the parameter f which is in units of seconds per nautical mile for a gauge of preserved dimensionality and unitless for a gauge of flattened dimensionality

$$h \overset{f}{\mapsto} h + f \qquad\qquad k \overset{e}{\mapsto} ke \qquad\qquad \begin{bmatrix} k & h \end{bmatrix} \overset{e,f}{\mapsto} \begin{bmatrix} ke & h + kf \end{bmatrix}$$

Composing time-on-distance or time-on-time gauge transformations is just a matter of adding or multiplying the respective parameter, where left-to-right order doesn't matter. There is an easy way to compose time-on-time-and-distance gauge transformations using 2×2 matrices.

10.2.8 The Need for Gauge Transformations

Gauge is rather abstractly defined. To say handicaps have the same gauge means that boats can race against each other using corrected time. But to say boats have a different gauge can only be quantified by a gauge transformation (or conversion) between them.

Gauge transformations can be necessary to merge performance handicaps for different fleets into a combined fleet. Or when boats cross into the jurisdiction of a different handicapping authority.

Fortunately, two fleets with different gauges can be brought to a common gauge by specifying a handicap in each gauge for a single common boat. If this is not possible a chain of connecting boats linking a chain of intermediate gauges can bring the fleets into alignment.

Measurement rules usually specify a single gauge by fiat and can ignore gauge transformations altogether. Unfortunately, there are many different measurement rules and converting between typically involves a gauge transformation. Likewise, converting between measurement and performance rules usually involves a gauge conversion or transformation.

10.3 The Invariance of Performance Lines with Respect to Gauge

10.3.1 Interpretation of Gauge Conversions as a Variable Substitution for q

We've presented gauge conversions and transformations as a way to represent exactly the same handicapping relationship with a different suite of handicaps, and have alluded to the necessarily different interpretation for the q variable with respect to the relative and absolute gauges. For time-on-distance or time-on-time there is a very simple way to understand a gauge conversion (or a simpler gauge transformation) as a simple shift or scaling of the q axis on the graphed performance line — this leads to a offset or scaling of the q variable and a countervailing negative offset or reciprocal scaling of the handicaps to effect a variable substitution that leaves the performance lines invariant under the conversion (or transformation). This is very easy to follow without an algebraic explanation. In this way a gauge conversion can be thought of as acting on handicaps, on one hand, and acting on the generalized parameter that is the q variable in an opposing fashion, on the other hand. Both viewpoints are equally valid.

Note that we are not considering all possible order preserving transformations of the q variable as we need to preserve the simple algebraic form of the handicapping relationship. A gauge conversion or transformation of the q variable must necessarily maintain the form of the *cap* function after the variable substitution. These correspond to the simplest possible relabelling of the q axis on the invariant graph of performance lines.

10.3.2 A Variable Substitution for Time-on-Time-and-Distance

For time-on-time-and-distance there is an equivalent offset and scaling of the q variable with a countervailing operation applied to the handicaps, but the algebraic relationship is a bit more complex

$$\begin{array}{ccc}
q \overset{F}{\mapsto} F + q & q \overset{E}{\mapsto} Eq & q \overset{E,F}{\mapsto} F + Eq \\
h \overset{f}{\mapsto} h + f & k \overset{e}{\mapsto} ke & \begin{bmatrix} k & h \end{bmatrix} \overset{e,f}{\mapsto} \begin{bmatrix} ke & h + kf \end{bmatrix} \\
F + f = 0 & Ee = 1 & Ee = 1 \text{ and } F + Ef = 0 = Fe + f
\end{array}$$

There is a very simple way to express this using 2×2 matrices which unifies the presentation for all these three styles of handicapping. We will explore this in a later chapter.

Chapter 11

Programming Corrected Times Without Rounding

11.1 Exact Arithmetic Using Rational Numbers

The best way to programme a computer to calculate and sort results is to use exact arithmetic using fractions (rational numbers). Many programming languages have such facilities built-in but, if not, explicit integer calculations are still possible. Python, C++, Haskell, Lisp, Perl, R, C# and many other programming languages can do exact rational arithmetic effortlessly. Java programmers have to use a clunky method call notation but the library functions are still available. Programmers of Excel and Javascript aren't so lucky. This chapter is written for them.

11.1.1 For the Unlucky

For sorting, we need only calculate the intermediate $\breve{u}$ from the corrected time formulae (we'll avoid the term *commensurable* in this context — it might be misleading). The magnitude of numbers seen is small enough that unreduced fractions will fit comfortably into 32 bit integers. Unlike what you learned in primary school, it never makes sense to reduce such a fraction into least terms.

We will ignore, for now, that our variables have units and dimensionality. Elapsed times are always entered to the closest second so t will be the whole number of seconds. The intermediate $\breve{u}$ will be a rational number whose units we don't care about, they will only be used for sorting. Course length can be represented as a rational number of nautical miles $d = \frac{L}{M}$ where L and M are whole numbers, M being preselected as the fineness of measurement — $M = 10$ would be a common choice for distances rounded to a tenth of a nautical mile.

Absolute gauge handicaps are whole numbers of seconds per nautical mile. For relative gauge handicaps we can consider the time coefficient as a whole number of seconds per kilosecond and the distance coefficient as an integer number of seconds per nautical mile. These give us k and h as integers.

11.1.2 Ordering Rational Numbers

Sorting in rational numbers is done by cross multiplying. For two boats (the one on the right distinguished with a $'$ prime) having intermediate terms (from the corrected time formulae) ready for sorting $\breve{u} = \frac{a}{b}$ and $\breve{u}' = \frac{a'}{b'}$ where a and a' are integers and b and b' are positive whole numbers then

$$\breve{u} < \breve{u}' \iff \frac{a}{b} < \frac{a'}{b'} \iff ab' < ba'$$

Equality works exactly the same way.

11.1.3 Rational Formulae with Integer Terms

Time-on-Distance	**Time-on-Time**	**Time-on-Time-and-Distance**
$\breve{u} = \dfrac{Mt - Lh}{M}$	$\breve{u} = \dfrac{t}{k}$	$\breve{u} = \dfrac{Mt - Lh}{Mk}$
where $L = Md$		where $L = Md$

11.1.4 Comparing the Handicapped Finish of Two Boats *Left & Right*

Let's consider ordering a pair of boats: the left boat with elapsed time t and handicapping factors k and h; the one on the right with t', k' and h' — L is course length in Mths of a nautical mile

for time-on-distance	$\breve{u} < \breve{u}'$	$\iff$	in integers	$Mt - Lh < Mt' - Lh'$
for time-on-time	$\breve{u} < \breve{u}'$	$\iff$	in integers	$tk' < kt'$
for time-on-time-and-distance	$\breve{u} < \breve{u}'$	$\iff$	in integers	$(Mt - Lh)k' < k(Mt' - Lh')$

The left boat beats the right boat with this integer comparison, would tie if the integer terms are equal, and would lose to the right boat if the ordering relation is reversed.

11.1.5 The Delta Between Two Boats *Left & Right*

Likewise we can write the difference between the left and right terms as $\Delta\breve{u}$

$$\Delta\breve{u} = \frac{Mt - Lh}{M} - \frac{Mt' - Lh'}{M} = \frac{M(t - t') - L(h - h')}{M}$$

$$\Delta\breve{u} = \frac{t}{k} - \frac{t'}{k'} = \frac{tk' - kt'}{kk'}$$

$$\Delta\breve{u} = \frac{Mt - Lh}{Mk} - \frac{Mt' - Lh'}{Mk'} = \frac{M(tk' - kt') - L(hk' - kh')}{Mkk'}$$

The left boat beats the right boat if $\Delta\breve{u}$ is positive, ties if zero and loses if negative. These expressions aren't used practically so we'll ignore the potential for 32 bit overflow. We do have to take overflow into account for the comparisons themselves.

11.1.6 Comparisons in a 32 bit Signed Integer or Floating Point

For time-on-distance or time-on-time 31 bit signed overflow isn't a concern. Time-on-time-and-distance will need to deal with the largest magnitudes. The k term will fit into 11 bits. The M mesh might take 5 bits. For a 31 bit signed comparison using cross multiplication that still leaves 15 bits for the elapsed time (nine hours) without having to worry about overflow. For programming languages that allow signed overflows to occur these limits would need to be enforced on inputs as triggering such an overflow would guarantee garbled results.

Any longer race can still be compared in 32 bits, but would first require a conversion to a mixed fraction; if the integer parts are still equal then cross-multiplying the fractional parts would be a bounded comparison of 22 bits (twice the 11 bits from the k term). 32 bit integer divisions are still blazingly fast so this is a complication for the programmer not the processor. Note that storing the mixed fraction would take three 32 bit words, one for the integral part, one for the fractional numerator and one for the fractional denominator.

Javascript uses a floating point representation for its integers leaving only 24 bits to work in. With cross multiplication, even time-on-time might lose precision in as short as a four hour race. Unlike 31 bit signed overflow this wouldn't lead to catastrophically wrong results, and indeed might never lead to an incorrect result, but defensive programming practices would lead us to always use a mixed fraction representation for the intermediate $\check{u}$. And any device that can support a web browser already has a floating point unit so speed isn't a concern.

11.1.7 Recapitulating Primary School

Let all variables be integers and consider the proper fraction $\frac{a}{b}$ where the numerator a can be any sign but the denominator b must be positive. We can use integer division of dividend a and divisor b to get quotient q and remainder r (sometimes called a *modulus*) yielding a mixed fraction $q;\frac{r}{b}$

$$\frac{a}{b} \text{ where } b > 0 \text{ using integer division } b\,\overset{q}{\overline{)\,a}}\,\underset{r}{} \implies \frac{a}{b} = q + \frac{r}{b} \text{ where } 0 \leqslant |r| < b$$

in Python	in Javascript	in Excel
`q, r = divmod(a, b)`	`r = a % b;`	QUOTIENT(a,b)
	`q = (a - r) / b;`	MOD(a,b)

These are largely similar, the only difference being that Javascript and Excel will return a remainder r with the opposite sign to Python when q is negative. In this situation Python will always return a positive remainder. Python has better support for integer arithmetic — generally doing what we want — but it also has unlimited precision rational numbers built into the language so we wouldn't actually need it.

11.1.8 Why Worry?

Why worry about low precision integers and rolling your own rational comparison when any decent programming language can do this for you? Javascript. A web page is more than powerful enough to score a race, but javascript is a primitive language requiring a lot of hand holding. And most phones are still 32 bit.

Having used exact arithmetic to place boats we could now fall back on floating-point arithmetic to display rounded corrected times, comfortable in the correctness of our results. Or we could soldier on to display precise results.

11.2 Reporting Corrected Time

We have only looked at the intermediate $\check{u}$ numbers so far. Mapping to the full corrected times $\check{\check{t}}$ involves, at most, a multiplication and an addition by integers so can not increase the magnitude of the denominator.

11.2.1 Corrected Time Between Two Boats *Left & Right*

The scratch boat has handicap $\left[{}^{\star}k \quad {}^{\star}h\right]$. As before the left boat has t, k and h and the right boat has t', k' and h'. Between them we have the $\Delta\check{u}$ from the previous chapter to derive the difference in corrected time $\Delta\check{\check{t}}$

$$\begin{aligned} \Delta\check{\check{t}} &= (\check{u} + {}^\bigstar hd) - (\check{u}' + {}^\bigstar hd) \\ &= \Delta\check{u} \\ &= \frac{M(t-t') - L(h-h')}{M} \end{aligned} \qquad \begin{aligned} \Delta\check{\check{t}} &= {}^\bigstar k\check{u} - {}^\bigstar k\check{u}' \\ &= {}^\bigstar k\Delta\check{u} \\ &= {}^\bigstar k\frac{tk' - kt'}{kk'} \end{aligned} \qquad \begin{aligned} \Delta\check{\check{t}} &= ({}^\bigstar k\check{u} + {}^\bigstar hd) - ({}^\bigstar k\check{u}' + {}^\bigstar hd) \\ &= {}^\bigstar k\Delta\check{u} \\ &= {}^\bigstar k\frac{M(tk' - kt') - L(hk' - kh')}{Mkk'} \end{aligned}$$

The time-on-distance difference is already as simple as it can get. For time-on-time and time-on-time-and-distance when the right boat is also the scratch boat then ${}^\bigstar k = k'$ so that

$$\Delta\check{\check{t}} = \frac{M\Delta t - L\Delta h}{M} \qquad \Delta\check{\check{t}} = \frac{tk' - kt'}{k} \qquad \Delta\check{\check{t}} = \frac{M(tk' - kt') - L(hk' - kh')}{Mk}$$

So differences in corrected time from the scratch boat have the same bounded denominator as corrected time itself. Again, these expansions are interesting but not terribly useful. If you are worrying about overflow you have already turned the $\check{u}$ into a mixed fraction

11.2.2 Ties on Corrected Time

Time-on-distance corrected times can be calculated directly as a whole number of Mths of a second for every boat so ties are M times less likely than they would be for rounded corrected times. In fact, tying is only possible in the case where $L(h-h')$ divided by M has remainder zero (is exactly divisible by M), in which case it is just as likely as for rounded corrected times, and is impossible for the other $M-1$ cases no matter what the elapsed times.

When the greatest common divisor of the time-on-time handicaps k and k' is large tying is no rarer than you would expect for rounded corrected times. But as the greatest common divisor becomes smaller the opportunities for tying become scarce. Broadly speaking exact ties are k times less likely than for ties on rounded corrected times. Time-on-time-and-distance combines both effects so ties are very unlikely.

11.2.3 Decimalization

Say you have displayed $\check{\check{t}}$ as a mixed fraction but competitors find it difficult to compare large fractions in their head. Let $\check{\check{t}}$ be $q;\frac{r}{b}$, a mixed fraction made out of whole number q, r and b equal to $q + \frac{r}{b}$ where $0 \leqslant r < b$. On an interactive page we can add decimals one-by-one on demand. Let the q_i be single decimal digits and r_n be natural numbers $0 \leqslant r_n < b$ in the notation $q.q_1q_2q_3\cdots q_n$ and $q.q_1q_2q_3\cdots q_n,\frac{r_n}{b}$ with the latter being an unusual but obvious generalization of mixed fraction notation. We can then express $\frac{10r_n}{b}$ as a mixed fraction $q_{n+1};\frac{r_{n+1}}{b}$ and append it to the decimal expansion $q.q_1q_2q_3\cdots q_n$ to get $q.q_1q_2q_3\cdots q_nq_{n+1},\frac{r_{n+1}}{b}$ equal to $q.q_1q_2q_3\cdots q_n,\frac{r_n}{b}$.

$$\begin{array}{lccc}
(q,r) = \operatorname{divmod}(a,b) & & & \dfrac{a}{b} \equiv q;\tfrac{r}{b} \\[1ex]
(q_1,r_1) = \operatorname{divmod}(10r,b) & \dfrac{10r}{b} = q_1;\tfrac{r_1}{b} & \implies & \dfrac{a}{b} \equiv q.\overbrace{q_1,\tfrac{r_1}{b}} \\[1ex]
(q_2,r_2) = \operatorname{divmod}(10r_1,b) & \dfrac{10r_1}{b} = q_2;\tfrac{r_2}{b} & \implies & \dfrac{a}{b} \equiv q.q_1\overbrace{q_2,\tfrac{r_2}{b}} \\[1ex]
(q_3,r_3) = \operatorname{divmod}(10r_2,b) & \dfrac{10r_2}{b} = q_3;\tfrac{r_3}{b} & \implies & \dfrac{a}{b} \equiv q.q_1q_2\overbrace{q_3,\tfrac{r_3}{b}}
\end{array}$$

$$\begin{array}{r@{\,}l}
 & \underline{\;q.q_1q_2q_3} \\
b\,) & a \\
 & \underline{\;*\;} \\
 & \;r.0 \\
 & \underline{\quad *\;} \\
 & \quad r_1 0 \\
 & \underline{\qquad *\;} \\
 & \qquad r_2 0 \\
 & \underline{\qquad\quad *} \\
 & \qquad\quad r_3
\end{array}$$

The sequence arises from long division as needed for a decimal quotient. In this way we can add decimals sufficient to visually compare corrected times yet also maintain exact precision.

Chapter 12

Positive-Sense versus Negative-Sense Handicaps

Many new handicapping authorities subscribe to the mistaken belief that the simplest possible corrected time formula leads to an easy to use handicapping system. This is quite backward.

12.1 The *chk* Function versus the *cap* Function

Recall that a handicap is best considered a relationship between either p and $\check{q}$ or between q and $\hat{p}$ formalized as either the *chk* or the *cap* function. Each boat will have its own mutually inverse *chk* and *cap* functions constrained by the style of handicapping employed. For time-on-distance and time-on-time handicapping this relationship can be parametrized by a single factor. For time-on-time-and-distance handicapping each boat will require two factors to specify the relationship. Performance curve scoring could get by with as few as three or four parameters without undue loss of generality.

The *chk*: $p \mapsto \check{q}$ function is the natural way to define corrected time through course-average pace

$$\mathrm{chk}^{\bigstar}(\check{\check{p}}) = \check{q} = \mathrm{chk}(p)$$

Whereas its inverse the *cap*: $\check{q} \mapsto p$ function naturally describes predicted pace with respect a general parameter q for the race

$$\hat{p} = \mathrm{cap}(q)$$

A Race Committee scoring a race uses the former whereas a competitor building a table of time allowances uses the latter. Time-on-distance, time-on-time and time-on-time-and-distance handicaps express a linear relationship between p and $\check{q}$. Linear functions are easily parametrized by a slope (time coefficient) and intercept (distance coefficient). So far as a competitor is concerned the only sensible parametrization of the handicapping relationship is through the *cap* function leading to the h, k and $\begin{bmatrix} k & h \end{bmatrix}$ handicaps we have already seen. Older rating and handicapping rules did things the sensible way, using a form of handicap most convenient for competitors. But many modern rules have abandoned good sense.

12.2 A Sign Convention for Handicapping

12.2.1 *Negative-Sense* Factors k, h in the Parametrization of *cap*: $\check{q} \mapsto p$ or $q \mapsto \hat{p}$

$$\begin{array}{llll}
\text{cap}(q) = h + q & \iff & \text{chk}(p) = p - h & \text{for time-on-distance} \\
\text{cap}(q) = kq & \iff & \text{chk}(p) = {}^{p}\!/\!{}_{k} & \text{for time-on-time} \\
\text{cap}(q) = h + kq & \iff & \text{chk}(p) = \frac{p-h}{k} & \text{for time-on-time-and-distance}
\end{array}$$

The *negative-sense* k and h factors are corrective, increasing for slower boats. These factors occur most naturally in the parametrization of the *cap* function.

12.2.2 *Positive-Sense* Factors b, c in the Parametrization of *chk*: $p \mapsto \check{q}$ or $\hat{p} \mapsto q$

$$\begin{array}{llll}
\text{chk}(p) = c + p & \iff & \text{cap}(q) = q - c & \text{for time-on-distance} \\
\text{chk}(p) = bp & \iff & \text{cap}(q) = {}^{q}\!/\!{}_{b} & \text{for time-on-time} \\
\text{chk}(p) = c + bp & \iff & \text{cap}(q) = \frac{q-c}{b} & \text{for time-on-time-and-distance}
\end{array}$$

These complementary *positive-sense* b and c factors are penalizing, increasing for faster boats. They occur most naturally in the parametrization of the *chk* function.

12.2.3 Other Parametrizations of the Handicapping Relationship Between $p \leftrightarrow \check{q}$

For other multi-factor relationships we can define the sign convention for each of its parameters depending on how they change with respect to slower or faster boats. The best parametrizations have a uniform sign convention for all factors. For a given context the relationship between p and $\check{q}$ or between q and $\hat{p}$ is best expressed as either the *cap* or the *chk* function corresponding to the negative-sense or the positive-sense respectively.

12.3 Switching Between Sign Conventions

12.3.1 With a Linear Handicapping Relationship

Compare the positive-sense handicaps in their application side-by-side with the equivalent negative-sense handicaps for time-on-distance, time-on-time and time-on-time-and-distance handicapping

$$\begin{array}{rcl}
\text{chk} : p \mapsto \check{q} & \text{cap} : \check{q} \mapsto p & \\
\hline
c + p = \check{q} = p - h & \check{q} - c = p = h + \check{q} & \text{for time-on-distance} \\
bp = \check{q} = {}^{p}\!/\!{}_{k} & {}^{\check{q}}\!/\!{}_{b} = p = k\check{q} & \text{for time-on-time} \\
c + bp = \check{q} = \frac{p-h}{k} & \frac{\check{q}-c}{b} = p = h + k\check{q} & \text{for time-on-time-and-distance}
\end{array}$$

to get the conversions and, more symmetrically but less conveniently, the invariant equations

$$\begin{array}{cccl}
\text{pos.} \leftarrow \text{neg.} & \text{invariant equations} & \text{pos.} \rightarrow \text{neg.} & \\
\hline
c = -h & c + h = 0 & -c = h & \text{for time-on-distance} \\
b = {}^{1}\!/\!{}_{k} & bk = 1 & {}^{1}\!/\!{}_{b} = k & \text{for time-on-time} \\
\left\{\begin{array}{c} b = {}^{1}\!/\!{}_{k} \\ c = -{}^{h}\!/\!{}_{k} \end{array}\right\} & \left\{\begin{array}{c} bk = 1 \\ c + bh = 0 = ck + h \end{array}\right\} & \left\{\begin{array}{c} {}^{1}\!/\!{}_{b} = k \\ -{}^{c}\!/\!{}_{b} = h \end{array}\right\} & \text{for time-on-time-and-distance}
\end{array}$$

Switching between the conventions is simple but, for applying on the water, handicaps expressed with a negative sense are much more useful. We will see this when we attempt to calculate time allowances from positive-sense handicaps.

12.3.2 Handicapping Schemes that Don't Admit a Linear Handicapping Relationship

Performance curve handicapping relies on a nonlinear *cap* function and there simply may not even exist a parametric expression of the *chk* function. If the performance curve is expressed piece-wise from linear segments then inverting each of the pieces is just as above as far as calculating the inverse is concerned — but no one would publish the piece-wise inverse as a *chk* function as the breaks between pieces would be determined on the $\check{q}$ axis. Simple second order polynomials can be inverted by the quadratic formula, but having to publish this as a canonical form is out of the question. Cubic splines have an analytic expression of their inverse but this is even uglier than the quadratic formula.

12.3.3 Units for Positive-Sense Handicaps

For gauges of preserved dimensionality (for all relative gauges and for all time-on-distance handicapping) positive-sense handicaps take the same units as their complementary negative-sense handicaps. Nevertheless the unitless k and b may be expressed as a ratio of time units which entails some unit conversion when switching between sign conventions. The negative-sense k is usually expressed in units of seconds per kilosecond $^{\mathrm{s}}/_{\mathrm{ks}}$ and for consistency with this presentation the positive-sense b should be expressed in the same manner; however, in practice, b is almost always expressed as a unitless multiplier to three decimal places. Converting from the negative-sense sign convention where k has units seconds per kilosecond $^{\mathrm{s}}/_{\mathrm{ks}}$

$$b = \frac{1000\,^{\mathrm{s}}/_{\mathrm{ks}} \times 1000\,^{\mathrm{s}}/_{\mathrm{ks}}}{k} \text{ (in } ^{\mathrm{s}}/_{\mathrm{ks}}) \qquad \text{or} \qquad b = \frac{1000\,^{\mathrm{s}}/_{\mathrm{ks}}}{k} \times 1.000 \text{ (without a unit)}$$

For gauges of flattened dimensionality (for all time-on-time and time-on-time-and-distance handicapping in an absolute gauge) the positive-sense time coefficient b is measured in units of speed and the matching positive-sense distance coefficient c is unitless. Units of speed are not familiar in handicaps. Knots or any power of ten thereof (as a measure of distance per *hour*) would require a seconds-to-hour conversion within the corrected time formula and is a poor choice for b. Thousandths of a nautical mile per kilosecond $^{\mathrm{mmi}}/_{\mathrm{ks}}$ is a better choice. Converting from the negative-sense sign convention where k and h both have units of seconds per mile $^{\mathrm{s}}/_{\mathrm{mi}}$

$$\begin{aligned} b &= \frac{1000\,^{\mathrm{s}}/_{\mathrm{ks}} \times 1000\,^{\mathrm{mmi}}/_{\mathrm{mi}}}{k} \text{ (in } ^{\mathrm{mmi}}/_{\mathrm{ks}}) \\ c &= -\frac{h}{k} \times 1000\,^{\mathrm{mmi}}/_{\mathrm{mi}} \end{aligned}$$

For time-on-time-and-distance the nominally unitless c is written as thousandths of a nautical mile per mile $^{\mathrm{mmi}}/_{\mathrm{mi}}$ (this SI styled unit may be read *millimile per mile* although that sounds a bit silly).

12.4 Defining Equations for Corrected Time in the Positive-Sense

From the general rule to the particular instances with respect to positive-sense handicaps we have

In General	on Distance	on Time	on Time and Distance
$\mathrm{chk}^{\bigstar}(\check{\check{p}}) = \mathrm{chk}(p)$	$\check{\check{t}} + {}^{\bigstar}cd = t + cd$	${}^{\bigstar}b\check{\check{t}} = bt$	${}^{\bigstar}b\check{\check{t}} + {}^{\bigstar}cd = bt + cd$

And in the relative gauge with ${}^{\bigstar}b_{\star} = 1$ and ${}^{\bigstar}c_{\star} = 0\,^{\mathrm{s}}/_{\mathrm{mi}}$ we get the simplest possible expression of

Time-on-Distance	**Time-on-Time**	**Time-on-Time-and-Distance**
$\check{\check{t}} = t + c_\star d$	$\check{\check{t}} = b_\star t$	$\check{\check{t}} = b_\star t + c_\star d$

The simplicity of these formulae is dangerously misleading; there is no good reason for them ever to be used. They offer nothing but complexity when used on the water.

12.5 Time Allowances and Differences in Handicapping Factors

Curiously, time-on-distance handicaps are always expressed in a negative-sense, despite that being the only style of handicapping for which time allowance calculations are just as easy with either sign convention.

For Δt being the time allowance between a pair of boats, Δk and Δh being the corresponding difference in their negative-sense handicaps and where the right-hand side of the proportionality in the unadorned t, k and h can refer to either boat

$$\begin{aligned} \Delta t : \Delta k \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t : k \times 1\,\mathrm{mi} && \text{for time-on-time} \\ \Delta t - \Delta h d : \Delta k \times 1\,\mathrm{mi} &\overset{\text{in proportion}}{=} t - hd : k \times 1\,\mathrm{mi} && \text{for time-on-time-and-distance} \end{aligned}$$

The proportionality can be easily extended by taking the ratio using the t, k and h for one selected boat and a ratio using the Δt, Δk and Δh for each competitor with respect to that one boat and equating them all in proportion. This is how we built time allowance tables using only the simplest arithmetic. Equations for a pair of boats with the corresponding difference in their positive-sense handicaps Δb and Δc suffer from a fatal flaw

$$(b + \Delta b)\Delta t + t\Delta b = 0 \qquad\qquad (b + \Delta b)\Delta t + t\Delta b + \Delta c d = 0$$

If t is the elapsed time of one of the boats then the $b+\Delta b$ term is the time coefficient of the handicap of the other boat, which leaves us with no efficient way to use these equations in fleet racing. The only way to efficiently compare yourself to several competitors simultaneously is to convert the positive-sense handicaps into negative-sense handicaps then use the proportions above.

The Δb and Δc can be useful, but only for the setting up of a pursuit race. For the benefit of competitors, handicaps should always be published with a negative sign convention.

12.6 Handicapping Pursuit Races and the Positive Sign Convention

12.6.1 Defining the Pursuit Race

Pursuit races handicap boats by penalizing them at their start rather than applying corrections at the finish line. Different course lengths and start times are chosen so that the finish order of boats determine standings directly. The handicapping we consider shall be time-on-distance, distance-on-distance (which uses time-on-time handicaps) and distance-and-time-on-distance (which uses time-on-time-and-distance handicaps). Distance-on-distance and distance-and-time-on-distance pursuit races have different course lengths for different boats — we can mentally model the race as having several starting lines (much like separate tee-offs in golf) and distance penalties would be applied at the start just as time penalties are (in practice we would have different intermediate marks for different classes).

For each type of boat, we write d° to denote the course length from the *pursuit specific start* and t° the time of the *pursuit specific starting signal* relative to a *nominal start*

Time-on-Distance	Distance-on-Distance	Distance-and-Time-on-Distance
$d^\circ = \lambda$	$d^\circ = b\lambda = \lambda/k$	$d^\circ = b\lambda = \lambda/k$
$t^\circ = c\lambda = -h\lambda$	$t^\circ = 0\,\mathrm{s}$	$t^\circ = c\lambda = {-h\lambda}/k$

Here λ is the *nominal course length* and only coincides with the actual course length in the time-on-distance case. For distance-on-distance and distance-and-time-on-distance we select a boat as scratch $\bigstar$, a course length for it ${}^\bigstar d$ and then calculate a nominal course length

$$\lambda = {}^\bigstar d/{}^\bigstar b = {}^\bigstar k \times {}^\bigstar d$$

This will be in units of distance using relative gauge handicaps and units of time when using absolute gauge handicaps; although the units make no difference in how it is used.

The natural sign convention for handicaps is positive and penalizing. Looking at differences in distances d° and times t° at the start we see that $\Delta d^\circ = \Delta b\lambda$ and $\Delta t^\circ = \Delta c\lambda$. For $\Delta b = b - {}^\bigstar b$ and $\Delta c = c - {}^\bigstar c$ we interpret the Δd° and Δt° as distance and time penalties with respect to the scratch boat. We can apply these Δd° and Δt° penalties at the start and ignore the full d° and t° thereafter.

Note that the nominal start and nominal course length should not be reported, they are merely intermediate steps in calculating the Δd° and Δt° with respect to the scratch boat's start. It makes sense to choose the first boat to start as scratch; this will be the boat with the lowest c factor in its positive-sense handicap.

12.6.2 Pursuit versus Corrected Time Handicaps

The *pursuit specific elapsed time* is $t - t^\circ$ where t is as elapsed time measured on the same clock as the the t°; that is, relative to the same nominal start. The course-average pace is the pursuit specific elapsed time over distance

$$p = \frac{t - t^\circ}{d^\circ}$$

Modeling boats as having a uniform speed over the course allows us to compare paces and corrected paces across courses of different length and directly relate distance-on-distance to time-on-time handicapping and distance-and-time-on-distance to time-on-time-and-distance handicapping

$$\check{q} = \frac{t - c\lambda}{\lambda} + c = \frac{t}{\lambda} \qquad \check{q} = b\frac{t}{b\lambda} = \frac{t}{\lambda} \qquad \check{q} = b\frac{t - c\lambda}{b\lambda} + c = \frac{t - c\lambda}{\lambda} + c = \frac{t}{\lambda}$$

Corrections applied to the computed average pace exactly cancel the handicapping penalty applied at the start so that corrected pace depends only on a boat's finish time and the fixed nominal course length — boats that finish at the same time have the same corrected pace. The same handicaps may be used for standard races with corrected elapsed times and pursuit races, the only difference being the most natural sign convention to express these handicaps.

And the calculations needed for a pursuit race are always done by the race organizers and distributed to competitors before racing, obviating the need for published positive-sense handicaps.

12.6.3 Better Units for Positive-Sense Handicaps in a Pursuit Race

For gauges of preserved dimensionality positive-sense handicaps take the same units as their complementary negative-sense handicaps. For pursuit races we will employ the factor c in computing a time penalty and the units s/mi are already appropriate for this usage. But we will want to reinterpret the unitless time coefficient b in terms of thousandths of a nautical mile per nautical mile mmi/mi — note that this is a synonym for the preferred unit ${}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}$ in corrected-time racing. When converting from k (with units ${}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}$) in the negative-sense sign convention

$$b = \frac{1000\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}} \times 1000\,{}^{\mathrm{mmi}}\!/\!{}_{\mathrm{mi}}}{k} \text{ (in } {}^{\mathrm{mmi}}\!/\!{}_{\mathrm{mi}})$$

We use this interpretation of the b factor to compute a distance penalty and never use it in a corrected time formula so calling it a *time coefficient* in this context and with these units in particular wouldn't be appropriate.

In a gauge of flattened dimensionality we already express b in units of thousandths of a nautical mile per kilosecond ${}^{\mathrm{mmi}}\!/\!{}_{\mathrm{ks}}$ so that the handicapping time coefficient b for the race on corrected time and the pursuit handicapping factor b are naturally expressed in identical units. We will reinterpret the unitless distance coefficient c which had been expressed in thousandths of a nautical mile per nautical mile ${}^{\mathrm{mmi}}\!/\!{}_{\mathrm{mi}}$ to be expressed in the units seconds per kilosecond ${}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}$ (a synonymous unit) so as to be best suited for computing a time penalty. When converting from the negative-sense sign convention

$$c = -\frac{h}{k} \times 1000\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}} \text{ (in } {}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}})$$

In this context and with these units it isn't fitting to call this handicapping factor a *distance coefficient*. Of course, using b as a time coefficient or c as a distance coefficient in a corrected time formula is something we should have discouraged anyway.

For pursuit races in either a gauge of preserved or flattened dimensionality, when expressing the units as fractions, we have thousandths of a nautical mile in the numerator for b, seconds in the numerator for c and a common unit of either distance or time in the denominator for both b and c; the units in the denominator will match the units for the nominal course length — miles or kiloseconds for a gauge of preserved or flattened dimensionality respectively. We achieve this by swapping out synonyms for *one one-thousandth* as needed to make it natural to reinterpret a time coefficient as the distance-on-distance component b of a handicap in a distance-on-distance or distance-and-time-on-distance pursuit race; likewise for a time-on-distance component c of a handicap in a time-on-distance or distance-and-time-on-distance pursuit race.

12.6.4 Example of a Distance-and-Time-on-Distance Pursuit Race

Let's rework a previous example between boats *Shindig* and *Hurricane*. We'll assume ${}^{\bigstar}d = 4\,\mathrm{mi}$. Shindig has a negative-sense handicap of $\begin{bmatrix}k & h\end{bmatrix} = \begin{bmatrix}858\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mi}} & 858\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mi}}\end{bmatrix}$ and is the scratch boat for the purposes of determining the nominal course length $\lambda = k\,{}^{\bigstar}d = 3432\,\mathrm{s} = 3.432\,\mathrm{ks}$. Converting $\begin{bmatrix}k & h\end{bmatrix}$ to a positive-sense handicap gives us $\begin{bmatrix}b & c\end{bmatrix} = \begin{bmatrix}1166\,{}^{\mathrm{mmi}}\!/\!{}_{\mathrm{ks}} & -1000\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}\end{bmatrix}$. Shindig's start is $d^{\circ} = {}^{\bigstar}d = 4\,\mathrm{mi}$ from the finish line.

Hurricane has a $\begin{bmatrix}672\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mi}} & 723\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{mi}}\end{bmatrix}$ negative-sense handicap. Its corresponding positive-sense handicap is $\begin{bmatrix}1488\,{}^{\mathrm{mmi}}\!/\!{}_{\mathrm{ks}} & -1076\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}\end{bmatrix}$ giving a difference of $\begin{bmatrix}\Delta b & \Delta c\end{bmatrix} = \begin{bmatrix}+322\,{}^{\mathrm{mi}}\!/\!{}_{\mathrm{ks}} & -76\,{}^{\mathrm{s}}\!/\!{}_{\mathrm{ks}}\end{bmatrix}$. Hurricane's start is $\Delta d^{\circ} = \Delta b \times 3.432\,\mathrm{ks} = 1105\,\mathrm{mmi} = 1.105\,\mathrm{mi}$ further from the finish and $\Delta t^{\circ} = \Delta c \times 3.432\,\mathrm{ks} = 261\,\mathrm{s} = 4\,\mathrm{min}\,21\,\mathrm{s}$ sooner.

Note that Hurricane starts sooner despite being a faster boat; its Δc penalty over Shindig is negative giving it an earlier start. Hurricane performs relatively well in light air. So in the 4 min 21 s it has before Shindig's start it will not have reduced the additional distance it has to sail in light air as much as it would have in heavy air.

12.6.5 The Distance-and-Time-on-Distance Pursuit Race's Raison d'Être

For a couple of boats that perform the same on average, the one that does better in light air will start earlier and further from the finish and the one that does better in heavy air will start later and nearer to the finish. At the time of the good-in-heavy-air boats' later start the earlier starter will have already passed by in heavy air, will have just caught up in medium air, and will still be behind in light air — the handicapping penalties applied at the start automatically adapt to suit the conditions.

12.6.6 The Same Example of Distance-and-Time-on-Distance in a Relative Gauge

Let's rework the same example between boats *Shindig* and *Hurricane* using relative gauge handicaps. Shindig has a negative-sense handicap of $\begin{bmatrix} k & h \end{bmatrix} = \begin{bmatrix} 1000\,{}^{\mathrm{s}}\!/\!_{\mathrm{ks}} & 0\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}} \end{bmatrix}$. And being scratch boat for the purposes of determining the nominal course length we also have $\lambda = 1.000 \times {}^{\bigstar}d = 4\,\mathrm{mi}$. This is a gauge of preserved dimensionality so the nominal course length λ is a distance. Converting $\begin{bmatrix} k & h \end{bmatrix}$ to a positive-sense handicap gives us $\begin{bmatrix} b & c \end{bmatrix} = \begin{bmatrix} 1000\,{}^{\mathrm{mmi}}\!/\!_{\mathrm{mi}} & 0\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}} \end{bmatrix}$. Shindig's start is $d^{\circ} = 4\,\mathrm{mi}$ from the finish line.

Hurricane has a $\begin{bmatrix} 785\,{}^{\mathrm{s}}\!/\!_{\mathrm{ks}} & +50\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}} \end{bmatrix}$ negative-sense handicap. Its corresponding positive-sense handicap is $\begin{bmatrix} 1275\,{}^{\mathrm{mmi}}\!/\!_{\mathrm{mi}} & -65\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}} \end{bmatrix}$ giving a difference of $\begin{bmatrix} \Delta b & \Delta c \end{bmatrix} = \begin{bmatrix} +275\,{}^{\mathrm{mmi}}\!/\!_{\mathrm{mi}} & -65\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}} \end{bmatrix}$. Hurricane's start is $\Delta d^{\circ} = \Delta b \times 4\,\mathrm{mi} = 1100\,\mathrm{mmi} = 1.1\,\mathrm{mi}$ further from the finish and $\Delta t^{\circ} = \Delta c \times 4\,\mathrm{mi} = 260\,\mathrm{s} = 4\,\mathrm{min}\,20\,\mathrm{s}$ sooner.

The numbers round just a little bit differently than before. This is not surprising and isn't a concern. There is no benefit to recalculating the pursuit specific d° and t° with regard to different scratch boats and no need for exact arithmetic.

12.7 Recapping the Roles of Negative and Positive Sense Handicaps

Handicaps expressed in a negative and positive sense are naturally suited to complementary roles

corrective (at finish)		penalizing (at start)	
h	time-on-distance	c	time-on-distance
k	time-on-time	b	distance-on-distance
$\begin{bmatrix} k & h \end{bmatrix}$	time-on-time-and-distance	$\begin{bmatrix} b & c \end{bmatrix}$	distance-and-time-on-distance

12.7.1 Distance Allowances in a Time-on-Time-and-Distance Race

Now it's easy to see how to calculate distance-and-time allowances in a time-on-time-and-distance corrected time race by re-imagining it as a distance-and-time-on-distance pursuit race. But distance allowances are, at best, a rule-of-thumb and cannot justify publishing positive-sense handicaps.

12.7.2 Horrible: "Corrected Time = Time Correction Factor × Elapsed Time"

A *time correction factor* (TCF) or, as IRC likes to call it, a *time correction coëfficient* (TCC) is a positive-sense relative-gauge time-on-time handicap for use in a simplified corrected time formula $\check{\check{t}} = \mathrm{TCF} \times t$ (in our variable naming convention the TCF would be $b_{\star}$ and the formula $\check{\check{t}} = b_{\star}t$). Both the ORR and ORC measurement rules, while still issuing a sensible general-purpose handicap on their rating certificates, now favour a TCF for club racing. But the sign convention is wrong for time-on-time racing — the TCF is really a distance-on-distance handicap being improperly applied. In contrast, a Portsmouth handicap DN, being a negative-sense relative-gauge time-on-time handicap ($k_\star \equiv \mathrm{DN}$) can be directly used to determine time allowances.

$$\Delta t : \Delta\mathrm{DN} \quad \overset{\text{in proportion}}{=} \quad t : {}^\bigstar\mathrm{DN}$$

And a PHRF handicap (a negative-sense relative-gauge time-on-distance handicap), while nominally time-on-distance, is often used for time-on-time by means of a simple gauge conversion. For example $\mathrm{PHRF} \mapsto k = \mathrm{PHRF} + 557\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}}$ is commonly used (cf. the general-purpose handicap). This k handicap can be used to much better effect than a TCF. Nicely $\Delta k = \Delta\mathrm{PHRF}$ giving a very easy to use proportionality

$$\Delta t : \Delta\mathrm{PHRF} \quad \overset{\text{in proportion}}{=} \quad t : (\mathrm{PHRF} + 557\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}})$$

12.7.3 Horrible: The Americap and ORC Performance Line

The TCF isn't the only burden we have to bear. Consider this abomination

"Corrected Time = PLT × Elapsed Time − PLD × Distance"

Americap introduced the two-factor handicap but its successor ORR has abandoned these efforts. Likewise ORC has dropped support for performance line scoring. The poor choice of parametrization couldn't have helped its popularity. There are so many things wrong with this parametrization that it is hard to know where to begin.

The positive sense PLT is a unitless number around 1.000 and the negative sense PLD is in seconds per nautical mile to one decimal place and apparently always small but positive in magnitude. The units at least are sensible. To start with we flip the sign for the PLD. Then the $\begin{bmatrix}\mathrm{PLT} & -\mathrm{PLD}\end{bmatrix}$ becomes a badly-centred relative-gauge positive-sense time-on-time-and-distance handicap. The $\mathrm{PLD} = 0\,{}^{\mathrm{s}}\!/\!_{\mathrm{mi}}$ would represent a boat too undercanvassed to race, so this gauge cannot be relative to an actual boat — the so-called corrected time in the formula is really just a commensurable time, used for comparison purposes only. We shall abandon this gauge and its ill-conceived simplified corrected time formula. Finding a boat with a middling PLD and a PLT close to 1.000 we can use it as a standard $\begin{bmatrix}{}^\bigstar\mathrm{PLT} & -{}^\bigstar\mathrm{PLD}\end{bmatrix}$ to rescue the relative gauge

$$\begin{bmatrix}b_\star & c_\star\end{bmatrix} = \begin{bmatrix}\frac{\mathrm{PLT}}{{}^\bigstar\mathrm{PLT}} & \frac{{}^\bigstar\mathrm{PLD}-\mathrm{PLD}}{{}^\bigstar\mathrm{PLT}}\end{bmatrix}$$

Now we have a sensible relative gauge handicap for a pursuit race and, from this, a corresponding negative sense handicap we might use in a regular handicapped race

$$\begin{bmatrix}k_\star & h_\star\end{bmatrix} = \begin{bmatrix}\frac{{}^\bigstar\mathrm{PLT}}{\mathrm{PLT}} & \frac{\mathrm{PLD}-{}^\bigstar\mathrm{PLD}}{\mathrm{PLT}}\end{bmatrix}$$

These PLT and PLD handicapping factors were determined from a VPP so we can estimate absolute gauge handicaps using the standard boat's accompanying ${}^\bigstar\mathrm{GPH}$ to get

$$\begin{bmatrix}k & h\end{bmatrix} = \begin{bmatrix}\frac{{}^\bigstar\mathrm{PLT}}{\mathrm{PLT}}\,{}^\bigstar\mathrm{GPH} & \frac{\mathrm{PLD}-{}^\bigstar\mathrm{PLD}}{\mathrm{PLT}} + \frac{{}^\bigstar\mathrm{PLT}}{\mathrm{PLT}}\,{}^\bigstar\mathrm{GPH}\end{bmatrix} = \begin{bmatrix}\frac{{}^\bigstar\mathrm{PLT}\,{}^\bigstar\mathrm{GPH}}{\mathrm{PLT}} & \frac{\mathrm{PLD}-{}^\bigstar\mathrm{PLD}+{}^\bigstar\mathrm{PLT}\,{}^\bigstar\mathrm{GPH}}{\mathrm{PLT}}\end{bmatrix}$$

Any of the measurement rules could have given us a usable negative-sense absolute-gauge two-factor handicap. But the false legitimacy of the TCF led to an unfortunate combination of time-on-time and time-on-distance handicapping — a parametrization which looked natural to Race Committees but was inappropriate for everyone else.

Chapter 13

Handicapping 2×2 Matrix Notation

2×2 matrices formalize the application and solving of 2×2 systems of linear equations. Matrices are ubiquitous in math and science and many online resources are available. Not only possessing a certain elegance, 2×2 matrices operations can be very convenient for handicappers who need to apply gauge conversions or transformations to time-on-time-and-distance handicaps. But for competitors themselves they offer little advantage over the formulae and equations we have already seen.

The 2×2 matrices we are interested in may be denoted concretely by a two-by-two grid or abstractly by a boldface variable (using both upper and lower case variable names). We will avoid variable names for 2×1 column and 1×2 row vectors by explicitly stating them only in component form; this keeps the notation simple and unambiguous.

13.1 2×2 Handicapping Matrices

13.1.1 The Negative-Sense 2×2 Handicapping Matrix H

Time-on-Distance

$$\mathbf{H} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\Delta\mathbf{H} = \begin{bmatrix} 0 & \Delta h \\ 0 & 0 \end{bmatrix}$$

Time-on-Time

$$\mathbf{H} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Delta\mathbf{H} = \begin{bmatrix} \Delta k & 0 \\ 0 & 0 \end{bmatrix}$$

Time-on-Time-and-Distance

$$\mathbf{H} = \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix}$$

$$\Delta\mathbf{H} = \begin{bmatrix} \Delta k & \Delta h \\ 0 & 0 \end{bmatrix}$$

Using this 2×2 *handicapping matrix* we can write the defining equations of corrected time in terms of the 2×1 *pace column vectors* formed from a time/distance or a pace/one pair

$$^{\star}\mathbf{H}^{-1}\begin{bmatrix} \check{\check{t}} \\ d \end{bmatrix} = \begin{bmatrix} \check{u} \\ d \end{bmatrix} = \mathbf{H}^{-1}\begin{bmatrix} t \\ d \end{bmatrix} \qquad\qquad ^{\star}\mathbf{H}^{-1}\begin{bmatrix} \check{\check{p}} \\ 1 \end{bmatrix} = \begin{bmatrix} \check{q} \\ 1 \end{bmatrix} = \mathbf{H}^{-1}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

The application of the handicap can be easily verified as being matrix multiplication of the matrix inverse of the 2×2 handicapping matrix on the left with the 2×1 pace column vector on the right for all three styles of handicapping. These matrices are particularly easy to invert

$$\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & -\frac{h}{k} \\ 0 & 1 \end{bmatrix}$$

Also note that the bottom row of any 2×2 handicap matrix or its inverse is always $\begin{bmatrix} 0 & 1 \end{bmatrix}$ and when such a matrix is applied to a pace column (a time/distance or pace/one pair) it preserves the value of the second component (d or 1 respectively).

Juggling the ${}^{\star}\mathbf{H}$ to turn the defining equations of corrected time into formulae gives us

$$\begin{bmatrix}\check{\check{t}}\\ d\end{bmatrix} = {}^{\star}\mathbf{H}\begin{bmatrix}\check{u}\\ d\end{bmatrix} = {}^{\star}\mathbf{H}\mathbf{H}^{-1}\begin{bmatrix}t\\ d\end{bmatrix} \qquad\qquad \begin{bmatrix}\check{\check{p}}\\ 1\end{bmatrix} = {}^{\star}\mathbf{H}\begin{bmatrix}\check{q}\\ 1\end{bmatrix} = {}^{\star}\mathbf{H}\mathbf{H}^{-1}\begin{bmatrix}p\\ 1\end{bmatrix}$$

13.1.2 The Handicapping Matrix H as a Prediction

The **H** matrix gives a very easy way to invert its action for the *chk* function and a very nice realization of the *cap* function for the generalized parameter q or through the related $u = qd$

$$\begin{bmatrix}\hat{p}\\ 1\end{bmatrix} = \mathbf{H}\begin{bmatrix}q\\ 1\end{bmatrix} \qquad \iff \qquad \begin{bmatrix}\hat{t}\\ d\end{bmatrix} = \mathbf{H}\begin{bmatrix}u\\ d\end{bmatrix}$$

Any two boats with handicaps that differ by $\Delta\mathbf{H}$ are predicted to have their elapsed times differ by Δt with the overall pace of the race as determined by the free variable u

$$\begin{bmatrix}\Delta t\\ 0\end{bmatrix} = \Delta\mathbf{H}\begin{bmatrix}u\\ d\end{bmatrix}$$

If we further constrain the u so that

$$\begin{bmatrix}t\\ d\end{bmatrix} = \mathbf{H}\begin{bmatrix}u\\ d\end{bmatrix}$$

where t and $\mathbf{H}$ is the elapsed time and handicap of the rightmost boat of the pair then the Δt will be the time allowance for the left with respect to the right at time t

$$\begin{bmatrix}\Delta t\\ 0\end{bmatrix} = (\Delta\mathbf{H})\mathbf{H}^{-1}\begin{bmatrix}t\\ d\end{bmatrix}$$

Using a $\Delta\mathbf{H}$ for each of your competitors and then incrementing the u in a linear sequence is just another way to build yourself a time allowance table

$$t \text{ where } \begin{bmatrix}t\\ d\end{bmatrix} = \mathbf{H}\begin{bmatrix}u\\ d\end{bmatrix} \qquad \& \qquad \cdots \quad \Delta t \text{ where } \begin{bmatrix}\Delta t\\ 0\end{bmatrix} = \Delta\mathbf{H}\begin{bmatrix}u\\ d\end{bmatrix} \quad \cdots$$

13.1.3 The General 2×2 Handicapping Operation in Three Styles

In the above demonstration, by ignoring the units, we described both absolute and relative gauge handicaps — more precisely gauges of both flattened and preserved dimensionality.

Time-on-Distance	**Time-on-Time or Distance-on-Distance**	**Time-on-Time-and-Distance or Distance-and-Time-on-Distance**
$\begin{bmatrix}1 & f\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}e & 0\\ 0 & 1\end{bmatrix}$ where e positive	$\begin{bmatrix}e & f\\ 0 & 1\end{bmatrix}$ where e positive

We can, in fact, describe both negative and positive sense handicaps, gauge conversions and transformations, corrected time and time allowance formulae and pursuit race formulae and account for sloppy Race Committee records with 2×2 matrices conforming to one of these three styles.

13.1.4 Multiplying Two Such 2×2 Matrices Together in The Three Styles

Time-on-Distance	Time-on-Time or Distance-on-Distance	Time-on-Time-and-Distance or Distance-and-Time-on-Distance
$\begin{bmatrix}1 & F\\0 & 1\end{bmatrix}\begin{bmatrix}1 & f\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & F+f\\0 & 1\end{bmatrix}$	$\begin{bmatrix}E & 0\\0 & 1\end{bmatrix}\begin{bmatrix}e & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}Ee & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}E & F\\0 & 1\end{bmatrix}\begin{bmatrix}e & f\\0 & 1\end{bmatrix}=\begin{bmatrix}Ee & F+Ef\\0 & 1\end{bmatrix}$

The time-on-distance style of 2×2 matrices are additive in their single parameter and the time-on-time style are multiplicative in their single parameter. With both these styles of matrices we can swap the left and right terms of the matrix product without changing the result — a property we call *commutativity*. The time-on-time-and-distance style of 2×2 matrices combine their two parameters in a more complex way that makes the matrix product dependent on preserving the left and right order of its terms — such matrices rarely *commute*.

The matrix product of two time-on-distance style of matrices is clearly still a time-on-distance style of matrix. The matrix product of two matrices of the time-on-time style is also still a matrix of the time-on-time style once you note that for both E and e positive then Ee must also be positive. And same holds for the matrix product of two time-on-time-and-distance style of matrices being a time-on-time-and-distance style of matrix.

13.1.5 Inverting Such a 2×2 Matrix in The Three Styles

Taking a matrix inverse of one these 2×2 matrices conforms to its own style of matrix with the observation that if the upper left component e is positive then the upper left component of the inverse $1/e$ is also positive.

$$\begin{bmatrix}1 & f\\0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & -f\\0 & 1\end{bmatrix}\qquad\begin{bmatrix}e & 0\\0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{e} & 0\\0 & 1\end{bmatrix}\qquad\begin{bmatrix}e & f\\0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{e} & -\frac{f}{e}\\0 & 1\end{bmatrix}$$

13.1.6 Units in the Three Styles of 2×2 Matrices

Multiplying the matrices requires the units be compatible with the defining formulae. There may also be requirements for units on the lower left component which is always zero. We've been careful with units up to now, so no issues will arise in using these 2×2 matrices.

The units for the components of such 2×2 matrices depend on whether we are in a gauge of flattened dimensionality (i.e. when $\mathbf{H}$ is an absolute gauge handicap in a time-on-time or time-on-time-and-distance style.) or a gauge of preserved dimensionality (i.e. when $\mathbf{H}$ is a relative gauge handicap or is an absolute gauge time-on-distance style handicap). We need not flesh this out until section 13.8 Painful Detail On Units and Dimensionality at the end of the chapter.

13.2 Converting or Transforming the Gauge of Handicap Matrix H

A handicapping matrix $\mathbf{H}$ will have its gauge converted or transformed with a multiplication on the right by a 2×2 matrix whose style matches $\mathbf{H}$.

13.2.1 Converting Absolute to Relative Gauge $\mathbf{H} \mapsto \mathbf{H}_\star$ Using Absolute ${}^\bigstar\mathbf{H}$

$$\mathbf{H} \mapsto \mathbf{H}_\star = \mathbf{H}\ {}^\bigstar\mathbf{H}^{-1}$$

We note that $\mathbf{H}_\star^{-1} = {}^\bigstar\mathbf{H}\mathbf{H}^{-1}$ so that for the consistent choice of scratch $\bigstar$

$$\begin{bmatrix} \check{\check{t}} \\ d \end{bmatrix} = {}^\bigstar\mathbf{H}\mathbf{H}^{-1} \begin{bmatrix} t \\ d \end{bmatrix} = \mathbf{H}_\star^{-1} \begin{bmatrix} t \\ d \end{bmatrix}$$

This is our simplified corrected time formula. Equivalently ${}^\bigstar\mathbf{H}_\star = \mathbf{I}$ the 2×2 identity matrix. Choosing the scratch boat so that its handicapping matrix is the 2×2 identity matrix leads to the simplified corrected time formula however you come to it.

13.2.2 Converting Relative to Absolute Gauge $\mathbf{H}_\star \mapsto \mathbf{H}$ Using Absolute ${}^\bigstar\mathbf{H}$

$$\mathbf{H}_\star \mapsto \mathbf{H} = \mathbf{H}_\star\, {}^\bigstar\mathbf{H}$$

13.2.3 Gauge Transformation f Acting on H

A gauge transformation $\mathbf{f}$ acts on $\mathbf{H}$ by matrix multiplication on the right

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_\bullet = \mathbf{H}\mathbf{f}$$

Where $\mathbf{f}$ is a 2×2 matrix whose style matches $\mathbf{H}$.

In a Gauge of Flattened Dimensionality all the components of $\mathbf{f}$ are unitless.

In a Gauge of Preserved Dimensionality the components of $\mathbf{f}$ have units identical to the corresponding units of $\mathbf{H}$.

13.2.4 Mapping the Handicap of a Singled Out Boat $\bigstar$ to Map the Entire Gauge

If we know the handicap of a singled out boat ${}^\bigstar\mathbf{H}$ and know what the handicap should map to ${}^\bigstar\mathbf{H}_\bullet$, we can easily determine the gauge transformation to map every boat in the gauge. In the abstract

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_\bullet = \mathbf{H}\mathbf{f}$$

In particular we require

$$\begin{aligned} {}^\bigstar\mathbf{H}_\bullet &= {}^\bigstar\mathbf{H}\mathbf{f} \\ {}^\bigstar\mathbf{H}^{-1}\, {}^\bigstar\mathbf{H}_\bullet &= {}^\bigstar\mathbf{H}^{-1}\, {}^\bigstar\mathbf{H}\mathbf{f} \\ {}^\bigstar\mathbf{H}^{-1}\, {}^\bigstar\mathbf{H}_\bullet &= \mathbf{f} \end{aligned}$$

So that

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_\bullet = \mathbf{H}\big({}^\bigstar\mathbf{H}^{-1}\, {}^\bigstar\mathbf{H}_\bullet\big)$$

This will work equally well for a gauge conversion or a gauge transformation. In short, any gauge can be easily mapped to any other gauge.

13.2.5 Verifying that the Gauge Doesn't Effect Corrected Times

Any gauge can be mapped to any other gauge by a multiplication on the right by a 2×2 matrix $\mathbf{f}$ of the appropriate style. So let the scratch be identified by a $\bigstar$ and let

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{Hf} \quad \text{so that in particular} \quad {}^\bigstar\mathbf{H} \overset{\mathbf{f}}{\mapsto} {}^\bigstar\mathbf{Hf}$$

Now $\mathbf{f}$ has an inverse $\mathbf{f}^{-1}$ (the existence of which is all this demonstration requires) and

$$({}^\bigstar\mathbf{Hf})(\mathbf{Hf})^{-1} = ({}^\bigstar\mathbf{Hf})(\mathbf{f}^{-1}\mathbf{H}^{-1}) = {}^\bigstar\mathbf{H}(\mathbf{ff}^{-1})\mathbf{H}^{-1} = {}^\bigstar\mathbf{HH}^{-1}$$

Comparing this to the corrected times formula

$$\begin{bmatrix}\check{\check{t}}\\ d\end{bmatrix} = {}^\bigstar\mathbf{HH}^{-1}\begin{bmatrix}t\\ d\end{bmatrix} \mapsto ({}^\bigstar\mathbf{Hf})(\mathbf{Hf})^{-1}\begin{bmatrix}t\\ d\end{bmatrix} = {}^\bigstar\mathbf{HH}^{-1}\begin{bmatrix}t\\ d\end{bmatrix} = \begin{bmatrix}\check{\check{t}}\\ d\end{bmatrix}$$

And we see that corrected time is defined independently of the gauge.

13.2.6 Confirming that the Gauge Doesn't Effect Ordering of Commensurable $\check{u}$

We also require of our handicaps $\mathbf{H}$ that the commensurable $\check{u}$ order boats the same in any gauge. Any gauge can be mapped to any other gauge $\mathbf{H} \mapsto \mathbf{Hf}$ by a multiplication on the right by a 2×2 matrix $\mathbf{f}$ of the appropriate style.

$$\begin{bmatrix}\check{u}\\ d\end{bmatrix} = \mathbf{H}^{-1}\begin{bmatrix}t\\ d\end{bmatrix} \quad \Longrightarrow \quad \big(\mathbf{Hf}\big)^{-1}\begin{bmatrix}t\\ d\end{bmatrix} = \mathbf{f}^{-1}\mathbf{H}^{-1}\begin{bmatrix}t\\ d\end{bmatrix} = \mathbf{f}^{-1}\begin{bmatrix}\check{u}\\ d\end{bmatrix} \qquad \text{so} \qquad \begin{bmatrix}\check{u}\\ d\end{bmatrix} \mapsto \mathbf{f}^{-1}\begin{bmatrix}\check{u}\\ d\end{bmatrix}$$

But the ordering of commensurable column vectors is preserved by a common multiplication on the left by a 2×2 matrix in any of the three styles; the $\mathbf{f}^{-1}$ is just one instance. We'll be overly nice and demonstrate this for the given $\mathbf{f}$ in the time-on-time-and-distance style of which subsumes the other two styles in this regard

$$\mathbf{f}^{-1}\begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} \mathbf{f}^{-1}\begin{bmatrix}\check{u}'\\ d\end{bmatrix} \iff \begin{bmatrix}e & f\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} \begin{bmatrix}e & f\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\check{u}'\\ d\end{bmatrix} \iff \begin{bmatrix}\frac{\check{u}-fd}{e}\\ d\end{bmatrix} \underset{=}{<} \begin{bmatrix}\frac{\check{u}'-fd}{e}\\ d\end{bmatrix}$$
$$\iff \frac{\check{u}-fd}{e} < \frac{\check{u}'-fd}{e} \overset{e>0}{\iff} \check{u}-fd < \check{u}'-fd \iff \check{u} < \check{u}' \iff \begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} \begin{bmatrix}\check{u}'\\ d\end{bmatrix}$$

It holds true for a general 2×2 matrix of the appropriate style just a little more cleanly, skipping the whole unnecessary inversion step. Which leads to …

13.2.7 Confirming that the Choice of Scratch Preserves Ordering of Corrected $\check{\check{t}}$

This is just a corollary of the previous demonstration. The ${}^\bigstar\mathbf{H}$ which balances the $\mathbf{H}^{-1}$ in the definition of corrected time for the scratch boat $\bigstar$ is common to all the boats in a division

$$\begin{bmatrix}\check{\check{t}}\\ d\end{bmatrix} = {}^\bigstar\mathbf{H}\begin{bmatrix}\check{u}\\ d\end{bmatrix} = {}^\bigstar\mathbf{HH}^{-1}\begin{bmatrix}t\\ d\end{bmatrix}$$

This ${}^\bigstar\mathbf{H}$ is just another 2×2 matrix of the appropriate style and therefore preserves the ordering of the commensurable $\check{u}$. To be needlessly thorough

$$
{}^\bigstar\mathbf{H}\begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} {}^\bigstar\mathbf{H}\begin{bmatrix}\check{u}'\\ d\end{bmatrix} \iff \begin{bmatrix}{}^\bigstar k & {}^\bigstar h\\ 0 & 1\end{bmatrix}\begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} \begin{bmatrix}{}^\bigstar k & {}^\bigstar h\\ 0 & 1\end{bmatrix}\begin{bmatrix}\check{u}'\\ d\end{bmatrix}
$$

$$
\iff {}^\bigstar k\check{u} + {}^\bigstar hd < {}^\bigstar k\check{u}' + {}^\bigstar hd \iff {}^\bigstar k\check{u} < {}^\bigstar k\check{u}' \overset{{}^\bigstar k>0}{\iff} \check{u} < \check{u}' \iff \begin{bmatrix}\check{u}\\ d\end{bmatrix} \underset{=}{<} \begin{bmatrix}\check{u}'\\ d\end{bmatrix}
$$

And none of this should be surprising. We are simply representing a style of order preserving linear function with which we built our definition of corrected time, our *chk* and *cap* functions and are representing them as matrices in order to broaden their application to gauge conversions and transformations.

13.3 Gauge Transformations or Conversions Applied to q Variable

We will use the $\mathbf{H}$ matrices to realize the *cap* function for the generalized parameter q

$$
\begin{bmatrix}\hat{p}\\ 1\end{bmatrix} = \mathbf{H}\begin{bmatrix}q\\ 1\end{bmatrix}
$$

Reiterating what we have said before, this gauge can be mapped to any other gauge by a multiplication on the right of all the $\mathbf{H}$ by a common 2×2 matrix $\mathbf{f}$ of the appropriate style so that the corresponding commensurable column vectors are mapped with a multiplication on left by the common $\mathbf{F} = \mathbf{f}^{-1}$ that preserves ordering between boats. Flipping our viewpoint around to consider $\mathbf{F}$ as a gauge transformation or conversion applied to the domain of q; more precisely, an order preserving variable substitution for q or a relabelling of the horizontal scale on a graph of performance lines,

$$
\begin{bmatrix}q\\ 1\end{bmatrix} \overset{\mathbf{F}}{\mapsto} \begin{bmatrix}{}^\bullet q\\ 1\end{bmatrix} = \mathbf{F}\begin{bmatrix}q\\ 1\end{bmatrix} \quad\iff\quad \begin{bmatrix}q\\ 1\end{bmatrix} = \mathbf{F}^{-1}\begin{bmatrix}{}^\bullet q\\ 1\end{bmatrix} \qquad \text{so that} \qquad \begin{bmatrix}\hat{p}\\ 1\end{bmatrix} = \mathbf{H}\begin{bmatrix}q\\ 1\end{bmatrix} = \mathbf{H}\mathbf{F}^{-1}\begin{bmatrix}{}^\bullet q\\ 1\end{bmatrix}
$$

So that mutually inverse matrices $\mathbf{F}\mathbf{f} = \mathbf{I}$ act on the variable $q \overset{\mathbf{F}}{\mapsto} {}^\bullet q$ that is the generalized parameter for our predictions, on the one hand, and the handicaps $\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_\bullet$ that realize our *cap* function, on the other hand, to complete our variable substitution

$$
q \overset{\mathbf{F}}{\mapsto} {}^\bullet q \;\text{ by }\; \begin{bmatrix}q\\ 1\end{bmatrix} \overset{\mathbf{F}}{\mapsto} \begin{bmatrix}{}^\bullet q\\ 1\end{bmatrix} = \mathbf{F}\begin{bmatrix}q\\ 1\end{bmatrix} \quad \text{and} \quad \mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_\bullet = \mathbf{H}\mathbf{f} \quad \text{give} \quad \mathbf{H}\begin{bmatrix}q\\ 1\end{bmatrix} = \begin{bmatrix}\hat{p}\\ 1\end{bmatrix} = \mathbf{H}_\bullet\begin{bmatrix}{}^\bullet q\\ 1\end{bmatrix}
$$

Time-on-Distance

$$
\begin{bmatrix}q\\ 1\end{bmatrix} \mapsto \begin{bmatrix}1 & F\\ 0 & 1\end{bmatrix}\begin{bmatrix}q\\ 1\end{bmatrix} = \begin{bmatrix}F+q\\ 1\end{bmatrix}
$$

Time-on-Time

$$
\begin{bmatrix}q\\ 1\end{bmatrix} \mapsto \begin{bmatrix}E & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q\\ 1\end{bmatrix} = \begin{bmatrix}Eq\\ 1\end{bmatrix}
$$

Time-on-Time-and-Distance

$$
\begin{bmatrix}q\\ 1\end{bmatrix} \mapsto \begin{bmatrix}E & F\\ 0 & 1\end{bmatrix}\begin{bmatrix}q\\ 1\end{bmatrix} = \begin{bmatrix}F+Eq\\ 1\end{bmatrix}
$$

13.4 Commutativity and Sloppy Record Keeping

Race Committees have been known to be sloppy in recording the start time of divisions in time-on-distance handicapped races, or the course length of time-on-time handicapped races. That these failures in record keeping cannot effect the how finishers are placed is related to the commutativity of the 2×2 matrices corresponding to the style of handicapping. We wont belabour the point but simply note that multiplying all uncorrected pace columns vector by a 2×2 matrix in the corresponding style of handicapping is related to our sloppy practices and, as we have already seen, multiplying the commensurable column vectors or corrected pace column vectors by the same 2×2 matrix preserves ordering.

So we can relax our data collection requirements needed to compare boats on corrected time. For time-on-distance handicapping we need not handicap elapsed times, but any time offset from it, including time-of-day, while preserving the ordering of corrected finish times. In the time-on-time group, elapsed time can be handicapped without regard to units and either pace or time can be interchanged so course distance is not needed. What is more, this relaxation of required information also applies to the computation of performance handicaps (in the same style of handicapping) provided you are happy to do so in a relative gauge and are not trying to determine absolute gauge handicaps.

The time-on-time-and-distance style of matrices lack commutativity and the boat dependent handicapping operation must be applied directly to the pace column vector — common information of start time and course distance cannot be factored out of the uncorrected pace column vector — any attempt to do so could alter the ordering of corrected finishes and corrupt the results.

But even in a style of handicapping which might allow for it, no one should be happy with such sloppy record keeping practices. A Race Committee should always record the start time of each division, its best determination of course length and its best estimate of wind strength (or any other observation which would relevant to any post-race handicapping analysis), no matter the style of handicapping, in order to properly fulfill its obligation to the class.

13.5 Positive-Sense versus Negative-Sense 2×2 Matrices

13.5.1 The Positive-Sense 2×2 Handicapping Matrix C

Time-on-Distance

$$\mathbf{C} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

Distance-on-Distance

$$\mathbf{C} = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$$

Distance-and-Time-on-Distance

$$\mathbf{C} = \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix}$$

Given a 1×2 *nominal course length row vector* $\begin{bmatrix} \lambda & 0 \end{bmatrix}$ (which has a zero as its second component) we can multiply on the right by this positive sense 2×2 *handicapping matrix* $\mathbf{C}$ to determine the 1×2 *pursuit specific distance and start time row vector*

$$\begin{bmatrix} d^\circ & t^\circ \end{bmatrix} = \begin{bmatrix} \lambda & 0 \end{bmatrix} \mathbf{C}$$

The application of the handicap can be easily verified for all three styles of pursuit races.

13.5.2 Running a Pursuit Race Using the 2×2 Matrix Differences ΔC

For actually running a pursuit race we use the differences in the positive sense 2×2 handicapping matrices $\Delta\mathbf{C}$

$$\begin{bmatrix} \Delta d^\circ & \Delta t^\circ \end{bmatrix} = \begin{bmatrix} \lambda & 0 \end{bmatrix} \Delta\mathbf{C}$$

Calculating our differences with respect to a boat with the smallest c (the upper right component of its $\mathbf{C}$ matrix) gives us differential start times $\Delta t^\circ \geq 0\,\mathrm{s}$ after the reference boat. Of course, in distance-on-distance races all boats start at the same time anyway. In a four-legged windward/leeward races adding additional *pursuit specific* windward marks at $\Delta d^\circ \div 4$ further to windward than the reference windward mark (negative Δd° implying the opposite direction) would satisfy our requirements for running a distance-on-distance or distance-and-time-on-distance race without too much difficulty.

13.5.3 Consistency Between Positive and Negative Sense Handicaps

This positive sense 2×2 handicapping matrix is just the matrix inverse of the negative sense **H** and, as such, doesn't really merit it's own definition. To give the **C** its own definition we must constrain the relation between the **H** and the **C** with the invariant equation $\mathbf{HC} = \mathbf{I}$ where **I** is the 2×2 identity matrix. Any gauge conversion or transformation applied to one must be complementarily applied to the other to preserve the invariant. This is quite easy

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{Hf} \qquad \text{and} \qquad \mathbf{C} \overset{\mathbf{F}}{\mapsto} \mathbf{FC} \qquad \text{where} \qquad \mathbf{fF} = \mathbf{I}$$

Note that the positive sense **C** is acted on its left side by the 2×2 matrix which is an inverse to the the matrix which acts on the right of the corresponding **H**.

The units just work out just as we have seen before so these 2×2 matrix multiplications can represent gauge conversions and transformations in all the variations we have already defined.

13.5.4 Realizing Consistency Between Positive and Negative Sense Gauges

$$\begin{array}{ccccc}
h \mapsto h + f & k \mapsto ke & \begin{bmatrix} k & h \end{bmatrix} \mapsto \begin{bmatrix} ke & h + kf \end{bmatrix} & \textit{negative sense} & \mathbf{H} \mapsto \mathbf{Hf} \\
c \mapsto F + c & b \mapsto Eb & \begin{bmatrix} b & c \end{bmatrix} \mapsto \begin{bmatrix} Eb & F + Ec \end{bmatrix} & \textit{positive sense} & \mathbf{C} \mapsto \mathbf{FC} \\
F + f = 0 & Ee = 1 & Ee = 1 \text{ and } F + Ef = 0 = Fe + f & \textit{for consistency} & \mathbf{Ff} = \mathbf{I} = \mathbf{fF}
\end{array}$$

The $e > 0$, f, $E > 0$ and F are the components of **f** and **F**, corresponding in the obvious way for the style of handicapping.

13.6 Realizing 2×2 Matrices in Time-on-Time-and-Distance Style

Only the time-on-time-and-distance style of matrices truly interest us. Let's realize some of these abstractions into 2×2 matrices of the time-on-time-and-distance style with explicit components. Using 2×2 matrices is an excellent way to avoid the silly mistakes we all make when juggling so many terms.

13.6.1 Converting an Absolute to a Relative Gauge Using an Absolute ★

$$\begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} k_\star & h_\star \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^\bigstar k & {}^\bigstar h \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{{}^\bigstar k} & -\frac{{}^\bigstar h}{{}^\bigstar k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{k}{{}^\bigstar k} & \frac{h\,{}^\bigstar k - k\,{}^\bigstar h}{{}^\bigstar k} \\ 0 & 1 \end{bmatrix}$$

13.6.2 Converting a Relative to an Absolute Gauge Using an Absolute ★

$$\begin{bmatrix} k_\star & h_\star \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_\star & h_\star \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^\bigstar k & {}^\bigstar h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_\star\,{}^\bigstar k & h_\star + k_\star\,{}^\bigstar h \\ 0 & 1 \end{bmatrix}$$

13.6.3 Mapping the Handicap of a Singled Out Boat ★ to Map the Entire Gauge

In particular we require

$$\begin{bmatrix} {}^{\star}k_{\bullet} & {}^{\star}h_{\bullet} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix} \mathbf{f}$$

$$\begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} {}^{\star}k_{\bullet} & {}^{\star}h_{\bullet} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix} \mathbf{f}$$

$$\begin{bmatrix} \frac{1}{{}^{\star}k} & -\frac{{}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{\star}k_{\bullet} & {}^{\star}h_{\bullet} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{f}$$

$$\begin{bmatrix} \frac{{}^{\star}k_{\bullet}}{{}^{\star}k} & \frac{{}^{\star}h_{\bullet} - {}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix} = \mathbf{f}$$

So that

$$\mathbf{H} \overset{\mathbf{f}}{\mapsto} \mathbf{H}_{\bullet} = \mathbf{H} \begin{bmatrix} \frac{{}^{\star}k_{\bullet}}{{}^{\star}k} & \frac{{}^{\star}h_{\bullet} - {}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix}$$

Or in components

$$\begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \overset{\mathbf{f}}{\mapsto} \begin{bmatrix} k_{\bullet} & h_{\bullet} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{{}^{\star}k_{\bullet}}{{}^{\star}k} & \frac{{}^{\star}h_{\bullet} - {}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k\frac{{}^{\star}k_{\bullet}}{{}^{\star}k} & h + k\frac{{}^{\star}h_{\bullet} - {}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix}$$

13.6.4 Realizing ${}^{\star}\mathbf{H}\mathbf{H}^{-1}$

We can multiply the 2×2 matrices ${}^{\star}\mathbf{H}$ and $\mathbf{H}^{-1}$ together to realize their combined action

$${}^{\star}\mathbf{H}\mathbf{H}^{-1} = \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{k} & -\frac{h}{k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{{}^{\star}k}{k} & \frac{{}^{\star}hk - {}^{\star}kh}{k} \\ 0 & 1 \end{bmatrix}$$

We didn't do this in our earlier presentation because it was too ugly and not truly relevant to performing the actual calculations. It isn't relevant here either but is interesting to compare it to the gauge conversions.

$${}^{\star}\mathbf{H}\mathbf{H}^{-1} \text{ has inverse } \mathbf{H}\,{}^{\star}\mathbf{H}^{-1}$$

$$\begin{bmatrix} \frac{{}^{\star}k}{k} & \frac{{}^{\star}hk - {}^{\star}kh}{k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix}^{-1} \text{ has inverse } \begin{bmatrix} k & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{\star}k & {}^{\star}h \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{k}{{}^{\star}k} & \frac{h\,{}^{\star}k - k\,{}^{\star}h}{{}^{\star}k} \\ 0 & 1 \end{bmatrix}$$

13.7 Realizing 2×2 Matrices for Distance-and-Time-on-Distance

To further familiarize ourselves with the nitty-gritty of 2×2 matrix calculations, we will realize the components of gauge conversions and transformations of the positive sense $\mathbf{C} \mapsto \mathbf{C}_{\bullet}$. Although it usually makes more sense to convert or transform the gauge of the negative sense $\mathbf{H}$ to get the $\mathbf{H}_{\bullet}$ and derive the $\mathbf{C}_{\bullet}$ from them. Indeed, if we are resorting to these mappings we have already made a very poor choice in handicapping system.

13.7.1 Converting an Absolute to a Relative Gauge Using an Absolute ★

$$\begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} b_\star & c_\star \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{{}^\bigstar b} & -\frac{{}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{b}{{}^\bigstar b} & \frac{c - {}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix}$$

13.7.2 Converting a Relative to an Absolute Gauge Using an Absolute ★

$$\begin{bmatrix} b_\star & c_\star \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_\star & c_\star \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^\bigstar b b_\star & {}^\bigstar c + {}^\bigstar b c_\star \\ 0 & 1 \end{bmatrix}$$

13.7.3 Mapping the Handicap of a Singled Out Boat ★ to Map the Entire Gauge

In particular we require

$$\begin{aligned}
\begin{bmatrix} {}^\bigstar b_\bullet & {}^\bigstar c_\bullet \\ 0 & 1 \end{bmatrix} &= \mathbf{F} \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} {}^\bigstar b_\bullet & {}^\bigstar c_\bullet \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix}^{-1} &= \mathbf{F} \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^\bigstar b & {}^\bigstar c \\ 0 & 1 \end{bmatrix}^{-1} \\
\begin{bmatrix} {}^\bigstar b_\bullet & {}^\bigstar c_\bullet \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{{}^\bigstar b} & -\frac{{}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix} &= \mathbf{F} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} \frac{{}^\bigstar b_\bullet}{{}^\bigstar b} & \frac{{}^\bigstar c_\bullet {}^\bigstar b - {}^\bigstar b_\bullet {}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix} &= \mathbf{F}
\end{aligned}$$

So that

$$\mathbf{C} \overset{\mathbf{F}}{\mapsto} \mathbf{C}_\bullet = \begin{bmatrix} \frac{{}^\bigstar b_\bullet}{{}^\bigstar b} & \frac{{}^\bigstar c_\bullet {}^\bigstar b - {}^\bigstar b_\bullet {}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix} \mathbf{C}$$

Or in components

$$\begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} \overset{\mathbf{F}}{\mapsto} \begin{bmatrix} b_\bullet & c_\bullet \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{{}^\bigstar b_\bullet}{{}^\bigstar b} & \frac{{}^\bigstar c_\bullet {}^\bigstar b - {}^\bigstar b_\bullet {}^\bigstar c}{{}^\bigstar b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{{}^\bigstar b_\bullet}{{}^\bigstar b} b & {}^\bigstar c_\bullet + \frac{{}^\bigstar b_\bullet}{{}^\bigstar b}(c - {}^\bigstar c) \\ 0 & 1 \end{bmatrix}$$

Despite going through these calculations, we would best avoid positive sense handicaps except as a derived form of the corresponding negative sense handicap.

13.8 Painful Detail On Units and Dimensionality

Consider 2×2 matrices whose components have their dimensionality annotated in brackets — [1] for numbers without a unit, $[L]$ for length, $[T]$ for time, $[LT^{-1}]$ for speed and $[L^{-1}T]$ for pace.

13.8.1 In Gauges of Preserved Dimensionality

… for all relative gauge handicaps as well as the absolute gauge for time-on-distance

The components of these 2×2 matrices have units as described

$$\begin{bmatrix} e[1] & f[L^{-1}T] \\ 0[LT^{-1}] & 1[1] \end{bmatrix} \text{ where } \begin{cases} e > 0 & \text{is unitless } [1] \\ f & \text{has dimensions of pace } [L^{-1}T] \end{cases}$$

These form a group of 2×2 matrices which includes all the gauge transformations and the **H** and **C** handicap matrices and their inverses (and hence their operation on the pace column vectors); it also includes all the gauge conversions between absolute and relative performance for the time-on-distance style of handicapping. In such a gauge the commensurable $\check{q}$ and $\check{u}$ have the same units and dimensionality as pace and elapsed time.

$$\begin{bmatrix} \check{\check{p}}[L^{-1}T] \\ 1[1] \end{bmatrix} \cdots \begin{bmatrix} \check{q}[L^{-1}T] \\ 1[1] \end{bmatrix} \cdots \begin{bmatrix} p[L^{-1}T] \\ 1[1] \end{bmatrix} \qquad\qquad \begin{bmatrix} \check{\check{t}}[T] \\ d[L] \end{bmatrix} \cdots \begin{bmatrix} \check{u}[T] \\ d[L] \end{bmatrix} \cdots \begin{bmatrix} t[T] \\ d[L] \end{bmatrix}$$

13.8.2 In Gauges of Flattened Dimensionality

… for the absolute gauge for time-on-time and time-on-time-and-distance handicaps

This scheme is more complicated. It requires a clear separation between the unitless gauge transformations, the early *chk-like* class-specific-handicapping operation (which operates on an uncorrected pace column vector to create a commensurable column vector) and the late balancing *cap-like* scratch-handicapping operation (which operates on an commensurable column vector to created a corrected pace column vector).

Consider the 2×2 matrices for the *chk-like* and *cap-like* operations and their direct realizations as the **C** and **H** handicap matrices which, being mutual inverses, must multiply together to become unitless

$$\underset{\textit{cap-like}\text{ dimensionality}}{\begin{bmatrix} k[L^{-1}T] & h[L^{-1}T] \\ 0[1] & 1[1] \end{bmatrix}} \qquad \underset{\textit{chk-like}\text{ dimensionality}}{\begin{bmatrix} b[LT^{-1}] & c[1] \\ 0[LT^{-1}] & 1[1] \end{bmatrix}} \qquad \begin{cases} k > 0 \text{ and } h & \text{dimensions of pace } [L^{-1}T] \\ b & \text{dimensions of speed } [LT^{-1}] \\ c & \text{unitless } [1] \end{cases}$$

These *chk-like* and *cap-like* operations also serve to convert between between gauges of flattened and preserved dimensionality. We use the $\star$ to distinguish the gauge of preserved dimensionality.

conversion	**H** negative-sense handicaps	**C** positive-sense handicaps
flattened $\mapsto$ preserved	$\mathbf{H} \mapsto \mathbf{H}_\star = \mathbf{H} \begin{bmatrix} \frac{1}{{}^\star k}[LT^{-1}] & -\frac{{}^\star h}{{}^\star k}[1] \\ 0[LT^{-1}] & 1[1] \end{bmatrix}$ with a *chk-like* action on the right	$\mathbf{C} \mapsto \mathbf{C}_\star = \begin{bmatrix} \frac{1}{{}^\star b}[L^{-1}T] & -\frac{{}^\star c}{{}^\star b}[L^{-1}T] \\ 0[1] & 1[1] \end{bmatrix} \mathbf{C}$ with a *cap-like* action on the left
preserved $\mapsto$ flattened	$\mathbf{H}_\star \mapsto \mathbf{H} = \mathbf{H}_\star \begin{bmatrix} {}^\star k[L^{-1}T] & {}^\star h[L^{-1}T] \\ 0[1] & 1[1] \end{bmatrix}$ with a *cap-like* action on the right	$\mathbf{C}_\star \mapsto \mathbf{C} = \begin{bmatrix} {}^\star b[LT^{-1}] & {}^\star c[1] \\ 0[LT^{-1}] & 1[1] \end{bmatrix} \mathbf{C}_\star$ with a *chk-like* action on the left

Next consider the unitless 2×2 matrices

$$\begin{bmatrix} e[1] & f[1] \\ 0[1] & 1[1] \end{bmatrix} \text{ where } e > 0 \text{ and } f \text{ are both unitless } [1]$$

They form a group of 2×2 matrices that could act on the commensurable $\check{q}$ and $\check{u}$ column vectors or can act as gauge transformations on the **C** or **H** matrices acting on the right or left respectively. In such a gauge the commensurable $\check{q}$ will be unitless and the commensurable $\check{u}$ will take units of distance. The corresponding commensurable column vectors will have the same dimensionality in both components.

$$\begin{bmatrix}\check{\check{p}}[L^{-1}T]\\ 1[1]\end{bmatrix}\ \cdots\ \begin{bmatrix}\check{q}[1]\\ 1[1]\end{bmatrix}\ \cdots\ \begin{bmatrix}p[L^{-1}T]\\ 1[1]\end{bmatrix} \qquad\qquad \begin{bmatrix}\check{\check{t}}[T]\\ d[L]\end{bmatrix}\ \cdots\ \begin{bmatrix}\check{u}[L]\\ d[L]\end{bmatrix}\ \cdots\ \begin{bmatrix}t[T]\\ d[L]\end{bmatrix}$$

Commensurable $\check{u}$ in units of distance looks odd. The absolute gauge gives a context for the $\check{u}$ on average but fails to do so in a any given race. Take, for example, a time-on-time handicapped race and another hypothetical race which is identical but runs at half the speed. All distances between boats shall be exactly as in the first race at the same point of completion (percentage-wise) but the $\check{u}$ shall be twice as large — any attempt to interpret these commensurable $\check{u}$ as actual distances is largely pointless. And, despite its perversity, it is not logically impossible to imagine a flattened absolute gauge for time-on-distance handicaps via an arbitrary choice of a common reference pace for all classes. The only sensible choice would be some sort of fleet average pace so that the $\check{u}$ in units of distance would be similar to those in the time-on-time and time-on-time-and-distance cases.

Chapter 14

Standard Units and Variables in those Units

14.1 In Systems of Units Used in This Book

Standard Units in Every Gauge

Dimensionality & Unit			Symbol	Usage
$[T]$	s	seconds	Δt°	pursuit specific start time penalty
			$\hat{t}, \Delta t$	predicted or critical time, time allowance
			$t, \check{\check{t}}$	elapsed time, corrected time
$[L]$	mi	miles	d	course length
	mmi	millimiles	Δd°	pursuit specific course distance penalty
$[L^{-1}T]$	s/mi	seconds per mile	$\hat{p}, \Delta p$	predicted or critical pace, pace allowance
			$p, \check{\check{p}}$	course-average pace, corrected pace
			h	handicap distance coefficient in negative-sense

Standard Units in a Gauge of Preserved Dimensionality

Dim.	Unit		Sym.	Usage
$[T]$	s	seconds	$\check{u}, u$	commensurable time, free variable akin to q
$[L]$	mi	miles	λ	pursuit-race nominal course length
$[L^{-1}T]$	s/mi	seconds per mile	$\check{q}, q$	commensurable pace, variable to *cap* function
				• relative gauge $\Longrightarrow$ standard, nominal pace
				• absolute gauge $\Longrightarrow$ T/D; ± s/mi around 0 s/mi
			c, h	h'cap distance coefficient in both sign conventions
			$c, \Delta c$	h'cap component of time in pursuit race
$[1]$		unitless	b, k	h'cap time coefficient in both sign conventions
	s/ks	seconds per kilosecond		thousandfold scaled for same and synonymous ...
$\equiv$	mmi/mi	millimiles per mile	$b, \Delta b$	... unit for h'cap component of distance in pursuit

Note that, in a gauge of flattened dimensionality, the commensurable time $\check{u}$ and pace $\check{q}$ are anomalous in not having units of time and pace respectively. There are no good choices for naming these quantities which are instantiations of the u or q free variables as derived from an observed elapsed time.

Standard Units in a Gauge of Flattened Dimensionality

Dim.	Unit		Sym.	Usage
$[T]$	ks	kiloseconds	λ	pursuit nominal course length
$[L]$	mi	miles	$\check{u}, u$	commensurable time, free variable akin to q
$[L^{-1}T]$	s/mi	seconds per mile	h, k	handicap coefficients in negative-sense
$[LT^{-1}]$	mi/ks	miles per kilosecond	b	h'cap time coefficient in positive-sense
	mmi/ks	millimiles per kilosecond		thousandfold scaled unit for same
			$b, \Delta b$	h'cap component of distance in pursuit race
$[1]$		unitless	c	h'cap distance coefficient in positive-sense
	mmi/mi	millimiles per mile		thousandfold scaled for same and synonymous ...
$\equiv$	s/ks	seconds per kilsecond	$c, \Delta c$	... unit for h'cap component of time in pursuit
	%	percent	$\check{q}, q$	commensurable pace, variable to *cap* function
				• T/T $\implies$ absolute gauge; ±% around 0%
				• T/TD $\implies$ absolute gauge; centred 100%

In the Tables Above

- Dimensionality is annotated by [1] for unitless, $[T]$ for time, $[L]$ for length, $[L^{-1}T]$ for pace and $[LT^{-1}]$ for speed;
- $[T]$ Kilosecond isn't a metric unit in normal usage. Should we have favoured sexagesimal handicaps we could have avoided the kilosecond and used an hour instead. As it stands racing authorities prefer to express ratings in thousandths which necessitates this peculiar unit.
- $[L]$ Miles are always nautical miles. Millimiles and its abbreviation mmi/mi is just a convenient way to express thousandths of a nautical mile, 1.852 m or roughly 6 ft, the height of a man. Had we favoured sexigesimally based numbers for our distance calculations we would have introduced a sixtieth of a sixtieth of a mile as a unit — the distance covered by a boat travelling one knot for one second — a convenient unit for racers but not worth exploring in this book.
- Delta variables (two character long variables staring with a Δ) are thought of as a difference of a pair of unadorned variables identified by stripping off the Δ and taking units to match the underlying variables. As such they can be used in different contexts not all of which are referenced in the tables above; the Δt and Δp will refer to their common usage as time and pace allowances.
- T/D, T/T and T/TD are abbreviations for time-on-distance, time-on-time and time-on-time-and-distance respectively. Then the abbreviations D/D and DT/D have the obvious interpretation for pursuit races. This shorthand notation is very common in discussions of handicapping — we've avoided these abbreviation elsewhere it in this book.

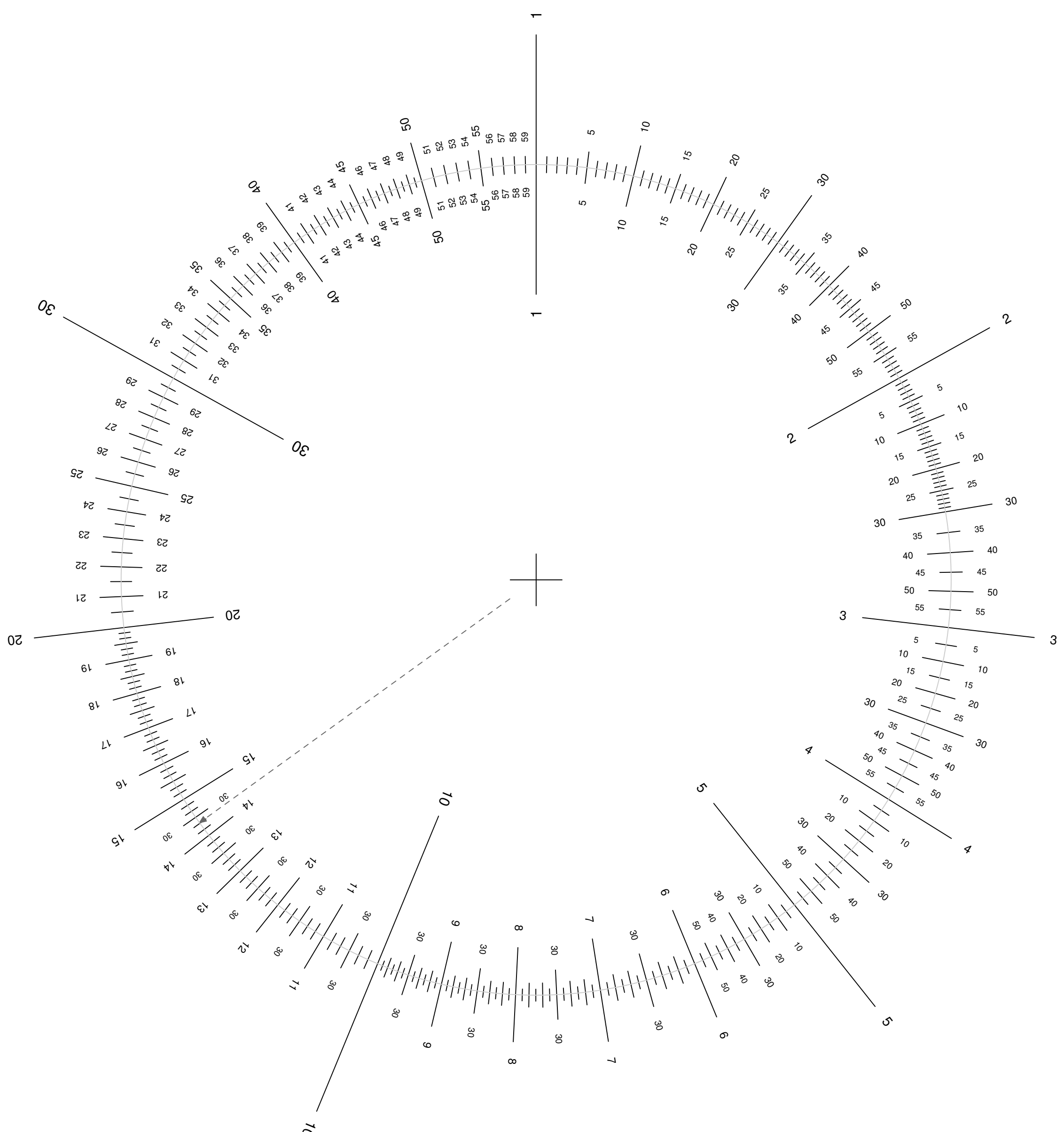
14.2 In Sexagesimal Systems of Units Not Used in This Book

Sexagesimal fractions are based on sixtieths called *minutes*. A sixtieth of a *minute* is called a *second minute* and so on. In common use we have the system of timekeeping based on hours (h), minutes (min) and seconds (s) and the system of measuring angles based on degrees (∘), arcminutes (′) and arcseconds (″). In the context of handicapping we would have a coherent system of units where

- the unit of time is the *hour*
- the unit of length is the *mile*
- the unit of speed is the *mile per hour* also known as the *knot*
- the unit of pace is the *hour per mile*
- unitless numbers may be *hour per hour* or *mile per mile*

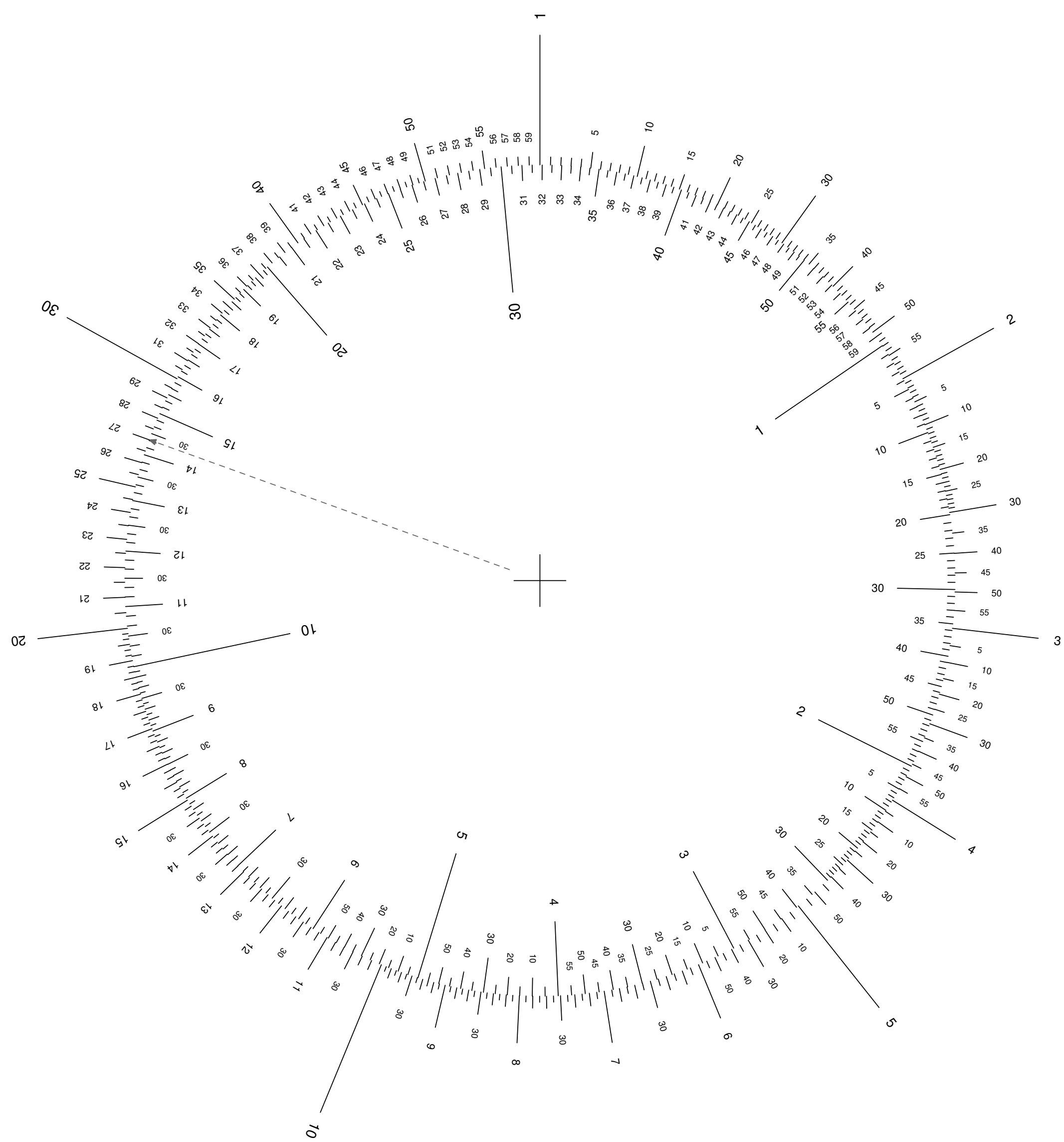
A whole number of hours and a whole number of miles can be subdivided with *minutes* and *second minutes* to express mixed fractions rather than using decimal fractions. A sixtieth of a mile is about 101 feet which may written sexagesimally as 0:01 mile. A boat travelling at 4 knots will cover 404 feet (written sexagesimally as 0:04 mile) in one minute (written sexagesimally as 0:01 hour). For handicapping we get readily expressed fractions well suited to mental arithmetic.

Most hand calculators support conversions between sexagesimal and decimal fractions for easy arithmetic. A base 60 circular slide rule makes calculations even easier. But slide rules are obsolete, sexagesimal fractions of a degree are obsolescent and hand calculators are now mere curiosities.



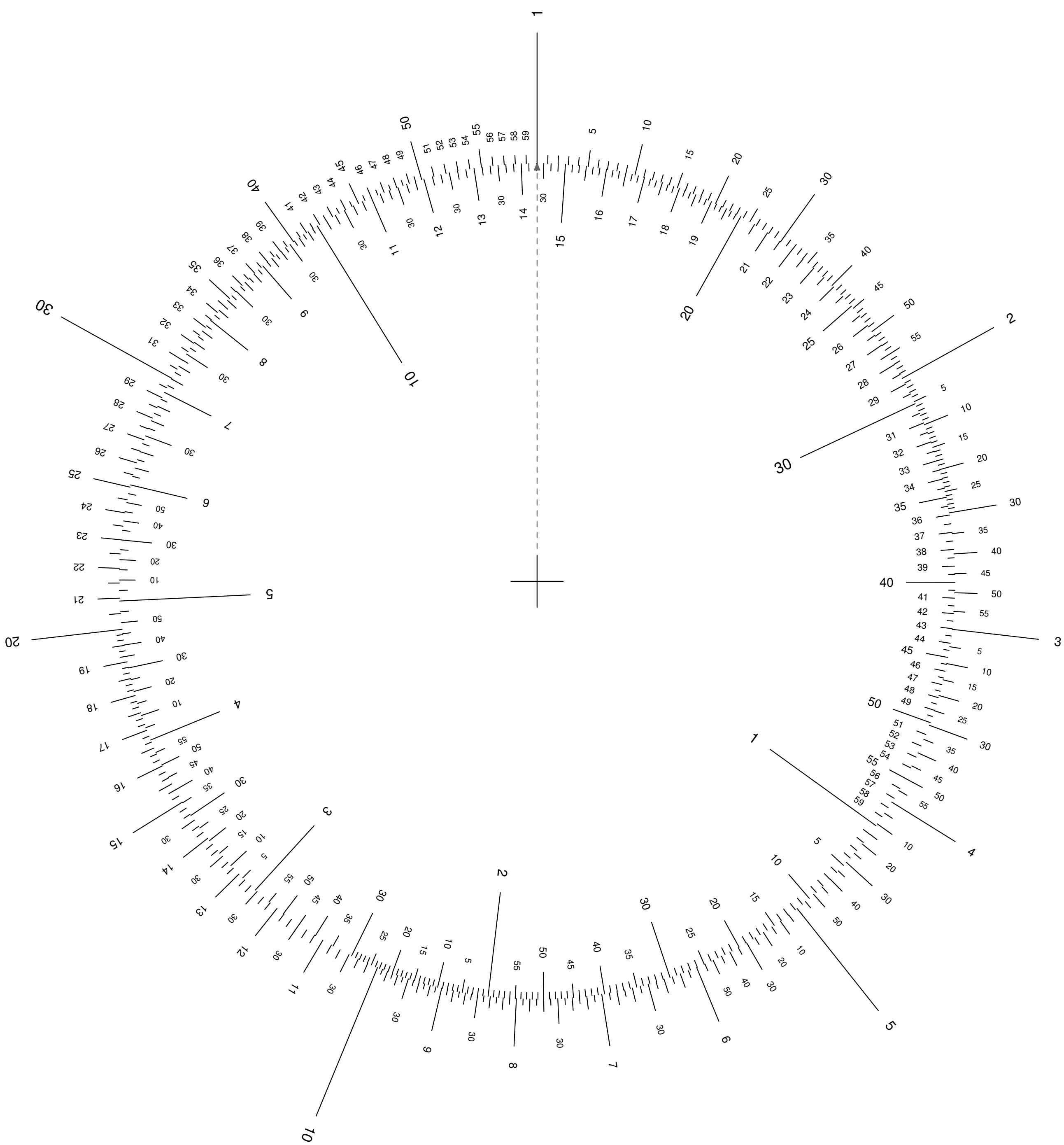
The slide rule consists of two graduated logarithmic scales which move relative to each other. Here, the outer dial has been aligned to the page. The cursor, an arrow which points from the centre, is adjustable but is fixed to the inner dial so that that the inner dial and the cursor rotate together. For your own use you will always fix the cursor to your own handicap, expressed sexagesimally, on the inner dial.

On all the examples above and below, we have set the cursor to point to 0:14:21 $^{\text{hour}}/_{\text{mile}}$ on the inner dial; this is *Shindig's* handicap g expressed sexagesimally. *Shindig's* can then pick a competitor with its Δg expressed sexagesimally; in a time-on-time race, using the cursor to rotate the inner dial to align her own handicap g on the inner scale with Δg on the outer scale, *Shindig* can now look up any interval of elapsed time t on the inner scale to determine the corresponding time allowance Δt on the outer.



Here *Shindig* is owed $-\Delta g = 0{:}00{:}27$ $^{\text{hour}}/_{\text{mile}}$ by her competitor *Mechanical Drone*. The inner dial has been rotated so that the cursor points to ten o'clock and the elapsed time and resulting allowance $t = 1\,\text{hour} \implies -\Delta t = 0{:}01{:}53\,\text{hour}$ can be read off at two o'clock. So the circular slide rule has been rotated for a given competitor and then can remain fixed throughout the race, reading around the inner scale for different elapsed times. If you wish to find time allowances for a different competitor the inner dial would need to be rotated as needed for each boat.

But there is another way to use the circular slide rule so that the inner dial rotates continuously throughout the race, with the cursor pointing towards elapsed time on the outer dial. At elapsed time $t = 1$ hour the cursor will be pointing straight up. With the rotation of the inner dial determined by the current time *Shindig* can look up Δg on the inner scale to read the time allowance Δt on the outer.



We can read off the time allowance:

at *12 o'clock* for	*Rhumb Punch*	$+\Delta g = 0{:}00{:}15\ ^{\text{hour}}/_{\text{mile}}$	$\Longrightarrow$	$+\Delta t = 0{:}01{:}03$ hour
at *2 o'clock* for	*Mechanical Drone*	$-\Delta g = 0{:}00{:}27\ ^{\text{hour}}/_{\text{mile}}$	$\Longrightarrow$	$-\Delta t = 0{:}01{:}53$ hour
at *4 o'clock* for	*Winged Elephant*	$-\Delta g = 0{:}00{:}51\ ^{\text{hour}}/_{\text{mile}}$	$\Longrightarrow$	$-\Delta t = 0{:}03{:}33$ hour
at *6 o'clock* for	*Hurricane*	$-\Delta g = 0{:}02{:}12\ ^{\text{hour}}/_{\text{mile}}$	$\Longrightarrow$	$-\Delta t = 0{:}09{:}12$ hour
at *7 o'clock* for	*Professor*	$+\Delta g = 0{:}00{:}03\ ^{\text{hour}}/_{\text{mile}}$	$\Longrightarrow$	$+\Delta t = 0{:}00{:}13$ hour

Part IV

Computing Performance Handicaps in Bulk

Chapter 15

General Concerns

In this part of the book, unless explicitly stated otherwise, we will always compute handicaps from data restricted to a single *league* (see below 15.1), we will always compute handicaps according to the *relative gauge criterion* (see below 16.2.3) and, for time-on-time-and-distance, we will always *seed the model* to obviate *degeneracy handling* (see below 16.2.4). Even when computing handicaps across multiple leagues we always remove from consideration classes which appear in no races and races which have no boats.

15.1 Handicapping Gauge and Performance Leagues

With gauge transformations, we can choose an arbitrary handicap for any one boat and then transform those of all the other boats to correspond to it without altering the effective handicapping in any way. We say such a choice determines the *gauge* for the handicaps. The gauge is common to all boats that race against each other, and by extension to other boats of the class and boats they have competed against using the handicap, and so on. For the purposes of performance handicapping we will say a class has *competed* against another class if we have finish data for a boat of the former class competing against a boat of the latter class. We will partition all the classes of boats into subsets — classes within each subset will be referred as the to the classes of a particular *league* and in loose usage the term *league* by itself may refer to the this subset. The partition is determined such that no class within one of the leagues has ever competed with a class in another league and, what is more, each of the subsets cannot be further partitioned. This doesn't mean any pair of the classes within an individual league have competed against each other directly but there must be a step-by-step path of direct competition linking them.

For those familiar with graph theory we note that partitioning into leagues can be defined in terms of the connected components of a bipartite graph which links classes to races, i.e. by following class to race to class to race and so on. Such components also partition races into subsets which are defined in one-to-one correspondence with the subsets of classes. We can call the former the *classes of the league* and the latter the *races of the league*. With less careful language the classes of the league could be named the *fleet* or just simply the *league*. And the races of the league could be called the *station* where it would not cause confusion.

Typically, we will restrict out attention to a single league; however, each new race has the potential to merge together previously separate leagues. Without prior finish data to compare across leagues pure performance handicapping for such a race is a seemingly impossible *bootstrapping* problem — but this is not completely true, as we will see below. On the other hand, brand new entrants of a class that has never raced before are, indeed, in a league of their own.

Each league will define its own gauge and therefore handicaps (and corrected times in new races) are not directly comparable across leagues. This presents a problem even in club racing where we fully expect there to be only a single league encompassing all the boats that have raced so far — new entrants will have to be seeded with a new handicap which makes sense in the existing gauge. A gauge transformation of handicaps may be necessary whenever leagues are merged together in order to create a common gauge.

15.2 Appropriate Statistical Methods and The Law of Large Numbers

Given the extent of performance handicapping in use and the potential for mass data collection it is clear that handicaps are best computed in toto using statistical models to match. The statistical models best suited are the most common: normal (Gaussian) errors in observed pace and simple regression models via the *cap* handicapping function. These lead to very straightforward calculations that are readily computerized with freely available software. These method compute statistics and mandate tests (the F-test in particular) to gauge their own efficacy.

By computing handicaps in toto we can be assured that no systemic errors are introduced into the handicaps by virtue of chaining statistical analysis using the output of an older analysis as the input to newer one in a deep chain of inference. Only error models based on the normal (a.k.a. Gaussian) probability distribution and regression methods which yield normally distributed estimates of the modelled handicaps avoid introducing systemic bias into computed handicaps when chaining inferences, i.e. computing a handicap for a new class of boat by adding estimated differences in performance between a pair of classes stepwise back to a standard boat. But even with normally distributed estimates it is better to return to the raw data for each subsequent analysis as statistical hypothesis testing is most easily interpreted in this context.

Computed handicaps are random variables, incorporating randomness from the underlying data set into their definition — they only estimate the unknowable parameters of the regression model. As such, each computed handicap has a readily quantified probability distribution inferred from the probability distribution of the collected data. As long as our statistical model is reasonable our inferred estimates will have very well understood properties. Variance is a measure of how *spread out* an estimate is — having as small a variance as is possible for an estimate is particularly important for small data sets. *Robust* statistics trade off other desirable properties in the probability distribution of the estimate for a low variance, especially in the case where the error model of the collected data deviates from a normal distribution.

The *percentile* is a robust statistic with a long history of use in handicapping. The 50^{th} *percentile*, 2^{nd} *quartile*, *median* and 0.5 *quantile* are all synonyms. They belong to a class of statistics related to the *empirical distribution function*. Multivariate statistics to compute a suite of handicaps in toto should be possible using empirical techniques, but it isn't clear how. While it is theoretically possible to compute good estimates (i.e. having a well understood probability distribution with desirable properties) from an inference chain of robust statistics in practice this would require using methods derived from Baysian statistics which are almost never computationally straightforward. In practice robust emperical statistics between pairs of boats have been added together to estimate handicaps for an entire fleet. This is not a valid statistical technique and the computed suite of handicaps will be poor.

Ironically, the simple average (arithmetic mean) which is a poor estimate for small data sets (especially where the error model deviates from a normal distribution) could have been used to compute time-on-distance handicaps via a stepwise inference boat to boat with a decent expectation of deriving a good suite of handicaps for a large fleet. The *law of large numbers* would ensure, as more data was added, such a system of handicaps (stated explicitly as normal distributed estimates each with a mean and variance) would improve predictably with means converging to a modelled ideal and with ever shrinking variances.

Lacking such a well quantified suite of handicaps it is best to use bulk data analysis to generate new handicaps from the raw data. Least-squares estimates and the accompanying standard errors and test statistics can be quickly computed using the free statistical package *R* or via bespoke code written in *Python* with the *numpy* and *scipy* libraries. For time-on-distance handicapping, a simple two-factor analysis of variance will suffice. The time-on-time and time-on-time-and-distance styles of handicapping require nonlinear regression models and non-linear solvers but are still well behaved and well supported by standard and freely available software.

Note that we have largely ignored the log transform (see section 16.5) which has traditionally be used to *linearize* the time-on-time style of handicapping. A statistical models which supports such a transformation is called a *general linear model* and is just as easy to deal with as a true linear model. So time-on-time handicapping can be as simple as time-on-distance as long as you understand the implications of the transformed error model. Indeed, the logarithmic error model for time-on-time may be more plausible than the normal error model.

15.3 Realized Variables in Handicapping Computations

15.3.1 Ad-Hoc Variables in Italics

Different boats will be indicated by greek superscripts $p^\alpha \triangleq t^\alpha \div d$, $t^\beta = p^\beta \times d$ and so on. In the computation of performance handicaps these abstract variables will be realized over different races distinguished by a latin subscript $p_r^\gamma \triangleq t_r^\gamma \div d_r$. As an alternative notation we may write ${}^\kappa p_r^i \equiv p_r^\alpha$ and ${}^\kappa t_r^i \equiv t_r^\alpha$ when α is the i^{th} boat of class κ to finish in race r. This works well in handicapping computations where it isn't necessary to distinguish between different boats within the class and agrees with our implied notation for handicapping factors ${}^\kappa k$, ${}^\kappa h$, ${}^\kappa b$ and ${}^\kappa c$ specified by class.

The barred variables ${}^\kappa \bar{p}_r$ and ${}^\kappa \bar{t}_r$ will be the average pace and average elapsed time of all boats of class κ to finish in race r. The all told averages within a race will be the doubly-barred $\bar{\bar{p}}_r$ and $\bar{\bar{t}}_r$. The average pace for all races isn't terrbily interesting but will denoted by the triply-barred $\bar{\bar{\bar{p}}}$ (see also subsection 16.3 about race-weighted averages).

15.3.2 Collected Observations in Boldface

Most computer code requires us to predetermine the arrays to hold the collected race data and the computed handicaps. This strict way of declaring the data structures *en masse* and then selecting individual entries from the whole can introduce clarity to our exposition. To this end we will refer to $\mathbf{t}$ and $\mathbf{d}$ for the time and distance observations (in statistics we still refer to this as a *sample* despite it containing all the data and not just a random sampling). From these data arrays we can *select* entries ${}^\kappa \mathbf{t}_r^i$ and $\mathbf{d}_r$ via predefined ranges of classes $\kappa \in \mathfrak{K}$ and races $r \in \mathfrak{R}$ and where the (potentially empty) valid range of i is dependent on the class κ and race r.

From the $\mathbf{t}$ and $\mathbf{d}$ we define $\mathbf{p}$ with comparable dimensions to the $\mathbf{t}$ such that ${}^\kappa \mathbf{p}_r^i = {}^\kappa \mathbf{t}_r^i \div \mathbf{d}_r$ when iterating over all κ, r and i. By declaring the average of zero boats to be zero we can define the reduced averages ${}^\kappa \overline{\mathbf{p}}_r$ and ${}^\kappa \overline{\mathbf{t}}_r$ for all classes and races. The doubly-barred $\overline{\overline{\mathbf{p}}}$ and $\overline{\overline{\mathbf{t}}}$ are arrays defined over all races. Likewise we define $\mathbf{q}$ which collects together imputed values for the q variable for each race. For example, denoting races as 0, 1, ... we have

$$\overline{\overline{\mathbf{p}}} = \begin{bmatrix} \overline{\overline{\mathbf{p}}}_0 & \overline{\overline{\mathbf{p}}}_1 & \cdots \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 & \cdots \end{bmatrix}$$

The use of superscripts to denote indices down a column and subscripts to denote indices out across a row is a fairly standard way to suggest the tabular layout of matrices. We will adopt this usage for its suggestive layout of listing boats and classes vertically and races horizontally while ignoring the theory behind it and without being bound to a consistent usage.

$$\mathbf{k} = \begin{bmatrix} {}^{\text{Buddy 24}}\mathbf{k} \\ {}^{\text{Frequency 24}}\mathbf{k} \\ {}^{\text{See in Sea 30}}\mathbf{k} \\ {}^{\text{Stone 22}}\mathbf{k} \\ {}^{\text{Chimera 33}}\mathbf{k} \\ \vdots \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} {}^{\text{Buddy 24}}\mathbf{h} \\ {}^{\text{Frequency 24}}\mathbf{h} \\ {}^{\text{See in Sea 30}}\mathbf{h} \\ {}^{\text{Stone 22}}\mathbf{h} \\ {}^{\text{Chimera 33}}\mathbf{h} \\ \vdots \end{bmatrix} \qquad \overline{\mathbf{p}} = \begin{bmatrix} {}^{\text{Buddy 24}}\overline{\mathbf{p}}_0 & {}^{\text{Buddy 24}}\overline{\mathbf{p}}_1 & \cdots \\ {}^{\text{Frequency 24}}\overline{\mathbf{p}}_0 & {}^{\text{Frequency 24}}\overline{\mathbf{p}}_1 & \cdots \\ {}^{\text{See in Sea 30}}\overline{\mathbf{p}}_0 & {}^{\text{See in Sea 30}}\overline{\mathbf{p}}_1 & \cdots \\ {}^{\text{Stone 22}}\overline{\mathbf{p}}_0 & {}^{\text{Stone 22}}\overline{\mathbf{p}}_1 & \cdots \\ {}^{\text{Chimera 33}}\overline{\mathbf{p}}_0 & {}^{\text{Chimera 33}}\overline{\mathbf{p}}_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

For time-on-time-and-distance the computed handicapping factors will be collected into two side-by-side arrays $\mathbf{k}$ and $\mathbf{h}$ for the time-on-time and time-on-distance components respectively. Per-class handicaps will be defined by packing together the individual handicapping factors $\begin{bmatrix} {}^\kappa\mathbf{k} & {}^\kappa\mathbf{h} \end{bmatrix}$. Likewise, for positive-sense handicaps we would have the arrays $\mathbf{b}$ and $\mathbf{c}$ with per-class handicaps ${}^\kappa\mathbf{c}$, ${}^\kappa\mathbf{b}$ and $\begin{bmatrix} {}^\kappa\mathbf{b} & {}^\kappa\mathbf{c} \end{bmatrix}$ depending on the style of handicapping.

15.4 Realized Handicapping Function Notation

It will be useful to have a symbolic notation for the realized handicapping operation covering any of the three styles of handicapping we are considering. For each class κ we will call the handicapping function chk^κ, its inverse cap^κ and the corresponding handicapping factors ${}^\kappa k$, ${}^\kappa h$, ${}^\kappa b$ and ${}^\kappa c$. As single argument functions the chk^κ and cap^κ are defined on pace p and the generalized parameter q analogous to the commensurable pace $\check{q}$.

$$\begin{aligned} {}^\kappa c + p &= \mathrm{chk}^\kappa(p) = p - {}^\kappa h & \qquad q - {}^\kappa c &= \mathrm{cap}^\kappa(q) = {}^\kappa h + q \\ {}^\kappa b p &= \mathrm{chk}^\kappa(p) = p \div {}^\kappa k & q \div {}^\kappa b &= \mathrm{cap}^\kappa(q) = {}^\kappa k q \\ {}^\kappa c + {}^\kappa b p &= \mathrm{chk}^\kappa(p) = (p - {}^\kappa h) \div {}^\kappa k & (q - {}^\kappa c) \div {}^\kappa b &= \mathrm{cap}^\kappa(q) = {}^\kappa h + {}^\kappa k q \end{aligned}$$

We define the commensurable $\check{q}$ for a boat β in its class κ or an enumerated boat i in class κ (and the corresponding average pace) in terms of this functional relationship,

$$\begin{aligned} \check{q}^\beta &= \mathrm{chk}^\kappa(p^\beta) & \qquad {}^\kappa\check{q}^i &= \mathrm{chk}^\kappa({}^\kappa p^i) & \qquad {}^\kappa\check{\bar{p}} &= \mathrm{chk}^\kappa({}^\kappa\bar{p}) \\ p^\beta &= \mathrm{cap}^\kappa(\check{q}^\beta) & {}^\kappa p^i &= \mathrm{cap}^\kappa({}^\kappa\check{q}^i) & {}^\kappa\bar{p} &= \mathrm{cap}^\kappa({}^\kappa\check{\bar{q}}) \end{aligned}$$

Should it be useful we can also define the elapsed time and commensurable $\check{u}$ by multiplying the corresponding pace and commensurable $\check{q}$ variants by course length. The function chk^κ is linear so within the class κ we can swap the order of taking the arithmetic mean of the class and applying a correction. The notation for the reduced commensurable pace ${}^\kappa\check{\bar{q}}$ is either the average of all the commensurable paces in the class or the chk^κ function applied to the class-average pace. And again, by linearity, the same is true for the reduced commensurable time ${}^\kappa\check{\bar{u}}$.

Within a race r the notation for the realized commensurable pace arrays is as you would expect

$$\begin{aligned} {}^\kappa\check{\mathbf{q}}_r^i &= \mathrm{chk}^\kappa({}^\kappa\mathbf{p}_r^i) & \qquad {}^\kappa\check{\overline{\mathbf{q}}}_r &= \mathrm{chk}^\kappa({}^\kappa\overline{\mathbf{p}}_r) \\ {}^\kappa\mathbf{p}_r^i &= \mathrm{cap}^\kappa({}^\kappa\check{\mathbf{q}}_r^i) & {}^\kappa\overline{\mathbf{p}}_r &= \mathrm{cap}^\kappa({}^\kappa\check{\overline{\mathbf{q}}}_r) \end{aligned}$$

Chapter 16

Using Least Squares to Compute Handicaps

16.1 Per-Race Cumulative Variables Dependent Upon Participation

Let $\mathbf{N}$ and $\mathbf{P}$ be arrays with the same dimensions as $\overline{\mathbf{p}}$. Letting the indices i implicitly vary over all the ordinals appropriate to number the finishers of class κ in race r we can denote their number ${}^{\kappa}\mathbf{N}_r = \sum_i 1$ and their cumulative pace ${}^{\kappa}\mathbf{P}_r = \sum_i {}^{\kappa}\mathbf{p}_r^i = {}^{\kappa}\mathbf{N}_r\, {}^{\kappa}\overline{\mathbf{p}}_r$ as sums over all such finishers. The average pace need not have been well-defined when ${}^{\kappa}\mathbf{N}_r$ was zero but it is convenient to have forced the corresponding reduced ${}^{\kappa}\overline{\mathbf{p}}_r$ to a value so we can write general expressions indexed over all κ and r — the actual value doesn't matter as they will only occur in terms which will be eliminated, but a placeholder value of zero is conventional.

Letting the κ implicitly vary over the classes $\kappa \in \mathfrak{K}$ and the races implicitly vary over the races $r \in \mathfrak{R}$ we can denote ${}^{\setminus}\mathbf{N}_r = \sum_\kappa {}^{\kappa}\mathbf{N}_r = \sum_{\kappa \in \mathfrak{K}} {}^{\kappa}\mathbf{N}_r$ and $\mathbf{N}_{\setminus} = \sum_r {}^{\kappa}\mathbf{N}_r = \sum_{r \in \mathfrak{R}} {}^{\kappa}\mathbf{N}_r$ as a shorthand. Avoiding the general notation for now, we will let $\mathbf{W}_r \triangleq {}^{\setminus}\mathbf{N}_r$ be the number of finishers in race r and let its counterpart ${}^{\kappa}\mathbf{M} \triangleq \mathbf{N}_{\setminus}$ be the total number of appearances of boats in the class κ. The doubly-barred $\overline{\overline{\mathbf{p}}}_r$ is the all told mean pace for race r so that

$$\mathbf{W}_r\, \overline{\overline{\mathbf{p}}}_r = \sum_\kappa {}^{\kappa}\mathbf{N}_r\, {}^{\kappa}\overline{\mathbf{p}}_r = \sum_\kappa {}^{\kappa}\mathbf{P}_r$$

Because the boats that actually compete from race to race are not always the same the $\overline{\overline{\mathbf{p}}}_r$ is almost never directly comparable between races; as a statistic $\overline{\overline{\mathbf{p}}}_r$ doesn't estimate anything of interest in the underlying performance model.

16.2 Regression and the Least-Squares Criterion

16.2.1 The Performance Index

Given the observed pace ${}^{\kappa}p_r^i$ we choose a regressed pace ${}^{\kappa}\hat{p}_r$ of the form $\mathrm{cap}^{\kappa}(q_r)$ where q_r runs free for each race r and the handicapping function runs free (according to the style of handicapping) for each class κ to minimize the performance index (a.k.a. the *cost function*)

$$Q = \sum_r \sum_\kappa \sum_i \left[{}^{\kappa}p_r^i - {}^{\kappa}\hat{p}_r\right]^2$$

The sum over the i varies over all the indices appropriate for the number of finishers in class κ and race r and sums to zero when when there are none.

In the equivalent but more precise array notation, given the observed pace array $\mathbf{p}$ we choose a regressed pace array $\widehat{\mathbf{p}}$ which is indexed over classes and races such that ${}^{\kappa}\widehat{\mathbf{p}}_r = \mathrm{cap}^{\kappa}(\mathbf{q}_r)$ for all classes κ and races r. The variable array $\mathbf{q}$ is indexed over races and runs free. For each class κ the variable handicapping function cap^{κ} runs free according to the style of handicapping. Given the freedom in $\mathbf{q}$ and all the cap^{κ} the $\widehat{\mathbf{p}}$ is chosen to minimize the performance index

$$Q = \sum_{r\in\mathfrak{R}} \sum_{\kappa\in\mathfrak{K}} \sum_{i=0}^{\mathrm{len}({}^{\kappa}\mathbf{p}_r)-1} \left[{}^{\kappa}\mathbf{p}_r^i - {}^{\kappa}\widehat{\mathbf{p}}_r\right]^2$$

16.2.2 Stationarity of the Performance Index

We write ∇Q for the gradient vector with respect to all the free variables in Q. The least squares perfomance index Q is minimized if and only if the stationarity condition $\nabla Q = \mathbf{0}$ holds. By the chain rule

$$\nabla Q = \sum_r \sum_\kappa \left(\sum_i 2\left[{}^{\kappa}\widehat{\mathbf{p}}_r - {}^{\kappa}\mathbf{p}_r^i\right] \right) \nabla\, {}^{\kappa}\widehat{\mathbf{p}}_r = 2 \sum_r \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\widehat{\mathbf{p}}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] \nabla\, {}^{\kappa}\widehat{\mathbf{p}}_r$$

Now $\nabla\, {}^{\kappa}\widehat{\mathbf{p}}_r$ as derived from the definition ${}^{\kappa}\widehat{\mathbf{p}}_r = \mathrm{cap}^{\kappa}(\mathbf{q}_r)$ are particularly simple

for Time-on-Distance	**for Time-on-Time**	**for Time-on-Time-and-Distance**
${}^{\kappa}\widehat{\mathbf{p}}_r = {}^{\kappa}\mathbf{h} + \mathbf{q}_r$	${}^{\kappa}\widehat{\mathbf{p}}_r = {}^{\kappa}\mathbf{k}\mathbf{q}_r$	${}^{\kappa}\widehat{\mathbf{p}}_r = {}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r$
$\dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, \mathbf{q}_r} = 1 \quad \dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\kappa}\mathbf{h}} = 1$	$\dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, \mathbf{q}_r} = {}^{\kappa}\mathbf{k} \quad \dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\kappa}\mathbf{k}} = \mathbf{q}_r$	$\dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, \mathbf{q}_r} = {}^{\kappa}\mathbf{k} \quad \dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\kappa}\mathbf{k}} = \mathbf{q}_r \quad \dfrac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\kappa}\mathbf{h}} = 1$

with No Interdependency Amongst Classes or Amongst Races for Any Style of Handicapping

$$s \neq r \implies \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, \mathbf{q}_s} = 0 \qquad \theta \neq \kappa \implies \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{k}} = 0 \qquad \theta \neq \kappa \implies \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{h}} = 0$$

So we can simplify and break out by free variable

$$0 = \frac{\partial Q}{\partial \mathbf{q}_s} = 2 \sum_r \sum_\kappa \left(\sum_i \left[{}^{\kappa}\widehat{\mathbf{p}}_r - {}^{\kappa}\mathbf{p}_r^i\right] \right) \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial \mathbf{q}_s} \implies 0 = \sum_\kappa {}^{\kappa}\mathbf{N}_s \left[{}^{\kappa}\widehat{\mathbf{p}}_s - {}^{\kappa}\overline{\mathbf{p}}_s\right] \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_s}{\partial \mathbf{q}_s}$$

$$0 = \frac{\partial Q}{\partial\, {}^{\theta}\mathbf{k}} = 2 \sum_r \sum_\kappa \left(\sum_i \left[{}^{\kappa}\widehat{\mathbf{p}}_r - {}^{\kappa}\mathbf{p}_r^i\right] \right) \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{k}} \implies 0 = \sum_r {}^{\theta}\mathbf{N}_r \left[{}^{\theta}\widehat{\mathbf{p}}_r - {}^{\theta}\overline{\mathbf{p}}_r\right] \frac{\partial\, {}^{\theta}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{k}}$$

$$0 = \frac{\partial Q}{\partial\, {}^{\theta}\mathbf{h}} = 2 \sum_r \sum_\kappa \left(\sum_i \left[{}^{\kappa}\widehat{\mathbf{p}}_r - {}^{\kappa}\mathbf{p}_r^i\right] \right) \frac{\partial\, {}^{\kappa}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{h}} \implies 0 = \sum_r {}^{\theta}\mathbf{N}_r \left[{}^{\theta}\widehat{\mathbf{p}}_r - {}^{\theta}\overline{\mathbf{p}}_r\right] \frac{\partial\, {}^{\theta}\widehat{\mathbf{p}}_r}{\partial\, {}^{\theta}\mathbf{h}}$$

This gives us the full set of stationarity equations in their most convenient form

Time-on-Distance	**Time-on-Time**	**Time-on-Time-and-Distance**
for each r	for each r	for each r
$0 = \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + \mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]$	$0 = \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] {}^{\kappa}\mathbf{k}$	$0 = \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] {}^{\kappa}\mathbf{k}$
for each κ	for each κ	for each κ
$0 = \sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + \mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]$	$0 = \sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] \mathbf{q}_r$	$0 = \sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] \mathbf{q}_r$
		$0 = \sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]$

Note that the stationarity conditions only need make use of the reduced array of average paces $\overline{\mathbf{p}}$ and does need access to the full $\mathbf{p}$.

16.2.3 A Fully Determined Regressed Pace

The regressed pace ${}^{\kappa}\widehat{\mathbf{p}}_r = \mathrm{cap}^{\kappa}(\mathbf{q}_r)$ estimates the "true" but unknowable model pace ${}^{\kappa}\mathfrak{p}_r$ of a corresponding form with an error model ${}^{\kappa}\mathbf{p}^i_r - {}^{\kappa}\mathfrak{p}_r$. The variable $\mathbf{q}_r$ added for each race is a sort of idealized commensurable pace against which all the boats compare and, in this context, is called the nominal pace. This least-squares criterion completely specifies the regressed pace ${}^{\kappa}\widehat{\mathbf{p}}_r$ insofar as the κ and r are relevant to the observations and for one-factor handicapping completely specifies ${}^{\kappa}\widehat{\mathbf{p}}_r$ even when no boat of class κ had competed in race r. This applies for regressions across multiple leagues even though each league's ${}^{\kappa}\widehat{\mathbf{p}}_r$ will end up being determined independently of the others — and classes that had never appeared and races that were not run must be omitted as not belonging to any league and being wholly irrelevant — even when we explicitly talk about regressions across multiple leagues we take the pruning of wholly irrelevant classes and races for granted. For multifactor handicaps the observations may not have sufficient information to completely constrain the shape of the cap^{κ} function for every class — so it may not be possible to extrapolate ${}^{\kappa}\widehat{\mathbf{p}}_r$ to every irrelevant κ and r. The more complete the data set the more likely that least squares will completely determine the ${}^{\kappa}\widehat{\mathbf{p}}_r$.

And, even where the ${}^{\kappa}\widehat{\mathbf{p}}_r$ is fully known, relevant and irrelevant alike, the parameters in the constraining $\mathrm{cap}^{\kappa}(\mathbf{q}_r)$ form are underspecified — the gauge runs free. Within each league there is gauge freedom embodied by a gauge transformation — i.e. with degrees of freedom equivalent to the number of factors needed to parameterize a single handicap. So for a single factor handicap in a single league that is just one degree of freedom. Further solving for the gauge freedom in the underlying parametric form $\mathrm{cap}^{\kappa}(\mathbf{q}_r) = {}^{\kappa}\widehat{\mathbf{p}}_r$ to minimize the sum of squares

$$Q^{\circledast} = \sum_r \sum_\kappa \sum_i \left[{}^{\kappa}\mathbf{p}^i_r - \mathbf{q}_r\right]^2$$

will uniquely specify the nominal pace $\mathbf{q}_r$ as being the *fleet standard pace* for the race and corresponding handicaps as belonging to the *fleet relative gauge.* We refer to this as the *relative gauge criterion.* Note that this further regression, like the main regression itself, can be performed on multiple leagues simultaneously but the results of doing so will be exactly the same as regressing each league independently. The only benefit of regressing multiple leagues simultaneously is that it unifies the error models and gives combined estimates of standard errors. But from a computational point-of-view we should always split the data into separate leagues for the initial computation even if we combine the error models afterwards.

For multifactor handicaps, lacking a fully determined ${}^{\kappa}\widehat{\mathbf{p}}_r = \mathrm{cap}^{\kappa}(\mathbf{q}_r)$ corresponds to not being able to specify all the handicapping factors independently. It is usually possible to determine all handicapping factors when the number of races for which a class has appeared exceeds the number of factors in the handicap but this does depend on the nominal paces being different for all such races. As nominal pace and handicaps are regressed simultaneously this is a nontrivial complication to the theory, but in practice is not a great issue.

16.2.4 Seeding the Model for Two-Factor Handicaps

Having underdetermined factors in the imputed handicaps corresponding to a regressed pace is not an insurmountable computational problem but it does highlight a difficulty where the predictive power of a multifactor handicap becomes poor in conditions away from the average. The obvious solution is to fall back to a simpler handicapping scheme for the boats in question. For example, a time-on-time-and-distance regression may be forced to degenerate to time-on-time handicaps for some boats. But, in the middle of a computation, it can difficult to detect where and when this will be necessary. Simply detecting impossible solutions is straightforward — but this wont tell us about improbable solutions where the determined handicap factors lack statistical significance.

Alternatively, we can seed the observed data to reduce the variance of all handicapping factors. Adding a nonsensical race $\circledcirc$ (here identified with a doughnut) to the set of races $\mathfrak{R}$ over which the $\mathbf{N}$ and $\overline{\mathbf{p}}$ are indexed can be remarkably effective for time-on-time-and-distance regressions

$$\begin{aligned} {}^{\kappa}\mathbf{N}_{\circledcirc} &= 1 && \text{for the full } \mathbf{p} \text{ each } {}^{\kappa}\mathbf{p}_{\circledcirc} \\ {}^{\kappa}\overline{\mathbf{p}}_{\circledcirc} &= 0 && \text{would be a singleton array} \end{aligned}$$

This weighs all the computed handicaps away from the potential extremes of a theoretical ideal towards a more sensible and only-marginally-less-precise happy medium. For time-on-time-and-distance handicaps within a single league, this has the convenient property of completely specifying the handicapping factors up to gauge freedom as determined by a two parameter gauge transformation — this is comparable in scope to the problem of regressing single factor handicaps.

As far as denoting the seeded data set we can either implicitly add columns of ones or zeroes to the $\mathbf{N}$ and $\overline{\mathbf{p}}$ and carry on without explicit notation to acknowledge the manipulation

$$\big[\mathbf{N} \mid \mathbf{j}_{\mathfrak{K}}\big] \to \mathbf{N} \qquad \big[\overline{\mathbf{p}} \mid \mathbf{0}_{\mathfrak{K}}\big] \to \overline{\mathbf{p}} \qquad \mathfrak{R} \cup \{\circledcirc\} \to \mathfrak{R} \ \ \text{(as an \textit{ordered} set)} \quad \Longrightarrow \quad \big[\mathbf{q} \mid q_{\circledcirc}\big] \to \mathbf{q}$$

Or we can tweak the performance index with one more free variable $q_{\circledcirc}$ independent of the unseeded free variables in the array $\mathbf{q}$. This introduces the seeded data as late as possible while maintaining the unseeded $\mathfrak{R}$, $\mathbf{N}$, $\overline{\mathbf{p}}$ (and $\mathbf{p}$) for

$$\begin{aligned} \underline{Q}_{\circledcirc} &= \sum_r \sum_\kappa {}^{\kappa}\mathbf{N}_r \big[{}^{\kappa}\overline{\mathbf{p}}_r - {}^{\kappa}\mathbf{h} - {}^{\kappa}\mathbf{k}\mathbf{q}_r\big]^2 + \sum_\kappa \big[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}q_{\circledcirc}\big]^2 \\ Q_{\circledcirc} &= \sum_r \sum_\kappa \sum_i \big[{}^{\kappa}\mathbf{p}^i_r - {}^{\kappa}\mathbf{h} - {}^{\kappa}\mathbf{k}\mathbf{q}_r\big]^2 + \sum_\kappa \big[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}q_{\circledcirc}\big]^2 \end{aligned}$$

Either way such fudged data will effect not only the computed handicaps but all other statistics resulting from the regression.

16.3 Degrees of Freedom versus Weighted Sums

16.3.1 Degrees of Freedom

Although not explicit in our presentation so far, we may define $\mathbf{W}_r$ and ${}^{\kappa}\mathbf{M}$ as weighted sums of the per-class per-race counts and not in a context where they need be natural numbers. The following degrees of freedom, denoted by ν, are always meant to be simple counts of entries. Let ν_p be the overall number of finishes, $\nu_{\overline{p}}$ the same but for classes of finishes and $\nu_{\overline{\overline{p}}}$ the number of races. Let ν_h be the number of different classes to have raced and hence the number of computable handicaps. For one-factor handicapping the number of degrees of freedom in the regressed pace is $\nu_{\hat{p}} = \nu_{\overline{\overline{p}}} + \nu_h - \nu_g$ where ν_g is the number of distinct leagues. For two-factor handicapping $\nu_{\overline{\overline{p}}} + \nu_h - \nu_g \leqslant \nu_{\hat{p}} \leqslant \nu_{\overline{\overline{p}}} + 2(\nu_h - \nu_g)$. Most classes, those with a non-degenerate computed handicap. will each contribute two degrees of freedom to the regressed pace. So the lower limit is unrealistically small and the upper limit is only reached if there are no degenerate handicaps computed — as expected for the seeded model. Also note that, in the seeded model, the seed data must be included in the counts.

Note that we always prune out irrelevant classes and races when we define our arrays so that $\mathfrak{R}$ has exactly $\nu_{\overline{\overline{p}}}$ elements and $\mathfrak{K}$ has exactly ν_h elements. This gives us a matrix representation of arrays so that: $\mathbf{N}$ and $\overline{\mathbf{p}}$ are both ν_h high by $\nu_{\overline{\overline{p}}}$ wide matrices; $\mathbf{M}$, $\mathbf{h}$ and $\mathbf{k}$ are ν_h high column vectors; $\mathbf{W}$, $\overline{\overline{\mathbf{p}}}$ and $\mathbf{q}$ are $\nu_{\overline{\overline{p}}}$ wide row vectors. From the predefined matrix of multiplicities $\mathbf{N}$ it is easy to compute the remaining degrees of freedom

$$\nu_p = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \qquad\qquad \nu_{\overline{p}} = \sum_r \sum_\kappa \begin{cases} 1 & {}^\kappa\mathbf{N}_r > 0 \\ 0 & {}^\kappa\mathbf{N}_r = 0 \end{cases}$$

ν_p	degrees of freedom in the full model $\mathbf{p}$
$\nu_{\overline{p}}$	degrees of freedom in the reduced model $\overline{\mathbf{p}}$
$\nu_{\overline{\overline{p}}} = \nu_q$	degrees of freedom in the nominal pace vector $\mathbf{q}$
ν_h	degrees of freedom in the handicap vector $\mathbf{h}$ or $\mathbf{k}$

16.3.2 Per-Class and Per-Race Weights

Sometimes we do not expect the regression model's error term ${}^\kappa\mathbf{p}^i_r - {}^\kappa\mathfrak{p}_r$ to be completely uniform over different classes and different races. Multiplying the expected variance by known per class weights ${}^\kappa\mathbf{m} > 0$ and per race weights $\mathbf{w}_r > 0$ to restore uniformity results in a weighted least squares model that is just as easy to solve and analyze as the unweighted model. The performance index for the weighted regression is

$$\widetilde{Q} = \sum_r \mathbf{w}_r \sum_\kappa {}^\kappa\mathbf{m} \sum_i \left[{}^\kappa\mathbf{p}^i_r - {}^\kappa\widehat{\mathbf{p}}_r\right]^2$$

And for the relative gauge criterion we have

$$\widetilde{Q}^{\circledast} = \sum_r \mathbf{w}_r \sum_\kappa {}^\kappa\mathbf{m} \sum_i \left[{}^\kappa\mathbf{p}^i_r - \mathbf{q}_r\right]^2$$

A reasonable value for the per race weights could be $\mathbf{w}_r = \mathbf{d}_r$ the course length for race r. Its harder to see a use for per class weights ${}^\kappa\mathbf{m}$ unless you already have a very good model of the variability in expected elapsed times broken down by class. For most usages we should set ${}^\kappa\mathbf{m} = 1$.

16.3.3 Weighted Sums and Averages

For the weighted model we will declare the weighted count and the weighted cumulative pace

$${}^\kappa\widetilde{\mathbf{N}}_r = {}^\kappa\mathbf{m} \cdot {}^\kappa\mathbf{N}_r \cdot \mathbf{w}_r \qquad\qquad {}^\kappa\widetilde{\mathbf{P}}_r = {}^\kappa\mathbf{m} \cdot {}^\kappa\mathbf{P}_r \cdot \mathbf{w}_r$$

Although the latter is redundant as we could derive it from the mean pace

$${}^\kappa\widetilde{\mathbf{P}}_r = {}^\kappa\widetilde{\mathbf{N}}_r\, {}^\kappa\overline{\mathbf{p}}_r$$

These are sufficient to define all the other weighted sums and weighted averages

$${}^\kappa\widetilde{\mathbf{M}} = \sum_r {}^\kappa\widetilde{\mathbf{N}}_r = {}^\kappa\mathbf{m}\Big(\sum_r {}^\kappa\mathbf{N}_r \cdot \mathbf{w}_r\Big) \qquad\qquad \widetilde{\mathbf{W}}_r = \sum_\kappa {}^\kappa\widetilde{\mathbf{N}}_r = \Big(\sum_\kappa {}^\kappa\mathbf{m} \cdot {}^\kappa\mathbf{N}_r\Big)\mathbf{w}_r$$

Not only are the ${}^\kappa\overline{\mathbf{p}}_r$ unaffected by these weights but so are the per-race $\overline{\overline{\mathbf{p}}}_r$ in the usual case where ${}^\kappa\mathbf{m} = 1$; but when the ${}^\kappa\mathbf{m} \neq 1$ the $\widetilde{\overline{\overline{\mathbf{p}}}}_r$ are still averages, just class-weighted averages

$$\widetilde{\mathbf{W}}_r \widetilde{\overline{\overline{\mathbf{p}}}}_r = \sum_\kappa {}^\kappa\widetilde{\mathbf{N}}_r\, {}^\kappa\overline{\mathbf{p}}_r = \sum_\kappa {}^\kappa\widetilde{\mathbf{P}}_r$$

We wont enforce the tilde notation for weighted models, we are simply using it here for clarity. In most cases we will want to use identical notation for weighted and unweighted models, particularly in the reduced model we are about to introduce; it is defined in terms of the $\mathbf{N}$ and $\overline{\mathbf{p}}$ so swapping in a weighted $\widetilde{\mathbf{N}}$ transforms it into a weighted model.

16.4 Sums-of-Squares and the Reduced Model

16.4.1 The Performance index and the Reduced Model

Q can be partitioned into a term $\mathrm{SS_W}$, the sum-of-squares-within-classes which is independent of the free variables, and a term $\underline{Q}$ which is the performance index for a reduced model which yields the same regressed pace and yet depends only on mean paces ${}^{\kappa}\overline{\mathbf{p}}_r$ weighted by their multiplicity ${}^{\kappa}\mathbf{N}_r$

$$\underbrace{\sum_r\sum_\kappa\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-\mathrm{cap}^{\kappa}(\mathbf{q}_r)\right]^2}_{Q}=\underbrace{\sum_r\sum_\kappa\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-{}^{\kappa}\overline{\mathbf{p}}_r\right]^2}_{\mathrm{SS_W}}+\underbrace{\sum_r\sum_\kappa{}^{\kappa}\mathbf{N}_r\left[{}^{\kappa}\mathbf{p}_r-\mathrm{cap}^{\kappa}(\mathbf{q}_r)\right]^2}_{\underline{Q}}$$

The residual sums-of-squares $\mathrm{SS_E}$ and $\underline{\mathrm{SS}}_\mathrm{E}$ are the minimum values of Q and $\underline{Q}$ achieved, respectively, so that $\mathrm{SS_E}=\mathrm{SS_W}+\underline{\mathrm{SS}}_\mathrm{E}$ with degrees of freedom $\nu_\mathrm{E}=\nu_p-\nu_{\hat{p}}$, $\nu_\mathrm{W}=\nu_p-\nu_{\bar{p}}$ and $\underline{\nu}_\mathrm{E}=\nu_{\bar{p}}-\nu_{\hat{p}}$. The sum-of-squares-between-classes $\mathrm{SS_B}$ with ν_B degrees of freedom is a synonym for $\underline{\mathrm{SS}}_\mathrm{E}$ with $\underline{\nu}_\mathrm{E}$ degrees of freedom so that $\mathrm{SS_E}=\mathrm{SS_W}+\mathrm{SS_B}$ with $\nu_\mathrm{E}=\nu_\mathrm{W}+\nu_\mathrm{B}$. Similarly, we define the all told sum-of-squares in each model $\mathrm{SS_A}$

$$\underbrace{\sum_r\sum_\kappa\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-\overline{\overline{\mathbf{p}}}_r\right]^2}_{\mathrm{SS_A}\text{ with }\nu_\mathrm{A}=\nu_p-\nu_{\bar{\bar{p}}}}=\underbrace{\sum_r\sum_\kappa\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-{}^{\kappa}\overline{\mathbf{p}}_r\right]^2}_{\mathrm{SS_W}}+\underbrace{\sum_r\sum_\kappa{}^{\kappa}\mathbf{N}_r\left[{}^{\kappa}\overline{\mathbf{p}}_r-\overline{\overline{\mathbf{p}}}_r\right]^2}_{\underline{\mathrm{SS}}_\mathrm{A}\text{ with }\underline{\nu}_\mathrm{A}=\nu_{\bar{p}}-\nu_{\bar{\bar{p}}}}$$

Note that we have no nice synonym for the $\underline{\mathrm{SS}}_\mathrm{A}=\mathrm{SS_A}-\mathrm{SS_W}$. Like expressions tend to proliferate. Any such sum-of-squares that does not rely on identifying individual boats within a class can be lowered into the reduced model by partitioning the full model sum-of-squares into the sum-of-squares-within-classes and the reduced model sum-of-squares.

The regression sum-of-squares $\mathrm{SS_R}$ with degrees of freedom $\nu_\mathrm{R}=\nu_{\hat{p}}-\nu_{\bar{\bar{p}}}$ is the same in both the full or reduced model. It is evaluated at the regressed pace from an expression which is equal in both the models

$$\sum_r\sum_\kappa\sum_i\left[\mathrm{cap}^{\kappa}(\mathbf{q}_r)-\overline{\overline{\mathbf{p}}}_r\right]^2=\sum_r\sum_\kappa{}^{\kappa}\mathbf{N}_r\left[\mathrm{cap}^{\kappa}(\mathbf{q}_r)-\overline{\overline{\mathbf{p}}}_r\right]^2$$

16.4.2 Weighted Regression Models and the Sum of Squares Within Classes

Consider this partitioning of the sum of squares associated with the weighted performance index

$$\overbrace{\sum_r\mathbf{w}_r\sum_\kappa{}^{\kappa}\mathbf{m}\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-\mathrm{cap}^{\kappa}(\mathbf{q}_r)\right]^2}^{\text{weighted model }\widetilde{Q}}$$
$$=\underbrace{\sum_r\mathbf{w}_r\sum_\kappa{}^{\kappa}\mathbf{m}\sum_i\left[{}^{\kappa}\mathbf{p}_r^i-{}^{\kappa}\overline{\mathbf{p}}_r\right]^2}_{\text{weighted model }\widetilde{\mathrm{SS}}_\mathrm{W}}+\underbrace{\sum_r\sum_\kappa{}^{\kappa}\widetilde{\mathbf{N}}_r\left[{}^{\kappa}\overline{\mathbf{p}}_r-\mathrm{cap}^{\kappa}(\mathbf{q}_r)\right]^2}_{\text{weighted reduced model }\underline{\widetilde{Q}}}$$

We can optimize within the weighted reduced model to determine the regressed pace. And the weighted reduced model differs from the unweighted reduced model only in the reduced sums $\widetilde{\mathbf{N}}$ versus $\mathbf{N}$. Simply by swapping in $\widetilde{\mathbf{N}}$ for $\mathbf{N}$ and $\widetilde{\mathrm{SS}}_{\mathrm{W}}$ for SS_{W} we can swap in the entire weighted regression model for the unweighted regression model without having to complicate our lives in any way. From here on we will treat the weighted regression model as just another special case in our presentation. And we will drop the use of the tilde with respect to the weighted model.

16.4.3 The Fleet Relative Gauge Criterion and a Further Reduced Model

With respect to the fleet relative gauge criterion, a partitioning of its sum-of-squares in both the full and reduced models reveals even more impressive simplifications

$$\underline{Q}^{\circledast} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\overline{\mathbf{p}}_r - \mathbf{q}_r\right]^2 = \underbrace{\sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\overline{\mathbf{p}}_r - \overline{\overline{\mathbf{p}}}_r\right]^2}_{\underline{\mathrm{SS}}_{\mathrm{A}}} + \underbrace{\sum_r \mathbf{W}_r \left[\overline{\overline{\mathbf{p}}}_r - \mathbf{q}_r\right]^2}_{\underline{\underline{Q}}^{\circledast}}$$

$$Q^{\circledast} = \sum_r \sum_\kappa \sum_i \left[{}^\kappa\overline{\mathbf{p}}_r - \mathbf{q}_r\right]^2 = \underbrace{\sum_r \sum_\kappa \sum_i \left[{}^\kappa\overline{\mathbf{p}}_r - \overline{\overline{\mathbf{p}}}_r\right]^2}_{\mathrm{SS}_{\mathrm{A}}} + \overbrace{\sum_r \mathbf{W}_r \left[\overline{\overline{\mathbf{p}}}_r - \mathbf{q}_r\right]^2}$$

$$Q^{\circledast} - \underline{Q}^{\circledast} = \mathrm{SS}_{\mathrm{W}} \qquad \underline{Q}^{\circledast} - \underline{\underline{Q}}^{\circledast} = \underline{\mathrm{SS}}_{\mathrm{A}} \qquad Q^{\circledast} - \underline{\underline{Q}}^{\circledast} = \mathrm{SS}_{\mathrm{A}}$$

$$\text{and between any two } \mathbf{q} \text{ and } {}^\star\mathbf{q} \quad \Delta Q^{\circledast} = \Delta\underline{Q}^{\circledast} = \Delta\underline{\underline{Q}}^{\circledast}$$

This gives us an equivalent and even further reduced model with a performance index to be minimized for the relative gauge criterion. Here the $\mathbf{q}_r$ cannot run free but are restricted to solutions corresponding to the regressed pace ${}^\kappa\widehat{\mathbf{p}}_r$

$$\underline{\underline{Q}}^{\circledast} = \sum_r \mathbf{W}_r \left[\overline{\overline{\mathbf{p}}}_r - \mathbf{q}_r\right]^2$$

16.4.4 Solving for the Fleet Relative Gauge Criterion given a Particular Solution

A gauge transformation maps a fixed particular solution to the regression model ${}^\star\mathbf{q}$ (a row vector of the ${}^\star\mathbf{q}_r$) componentwise to a variable general solution $\mathbf{q}$ via the gauge transformation parameters. To efficiently apply the gauge criterion to these general solutions it will be convenient to define some additional notation around *moments* which integrate powers of $\mathbf{q}_r$ with the sums

$$\mathfrak{W} = \sum_r \mathbf{W}_r \qquad\qquad \mathfrak{P} = \sum_r \overline{\overline{\mathbf{p}}}_r \mathbf{W}_r$$

We will subscript these $\mathfrak{W}$ and $\mathfrak{P}$ with bullets to indicate the power of $\mathbf{q}_r$, (zero for constants, one for *first order moments* and two for *second order moments*). With regard to this general solution we can find the minimum to this relative gauge criterion sum of squares $Q^{\circledast}$ via a stationarity equation for

$Q^{\circledast}$, $\underline{Q}^{\circledast}$ or $\underline{\underline{Q}}^{\circledast}$ evaluated at $\mathbf{q}$ with respect to each gauge transformation parameter G

$$0 = \frac{\partial Q^{\circledast}}{\partial G} = \frac{\partial \underline{Q}^{\circledast}}{\partial G} = \frac{\partial \underline{\underline{Q}}^{\circledast}}{\partial G} = 2\sum_r \mathbf{W}_r \left(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\right) \frac{\partial \mathbf{q}_r}{\partial G}$$

Time-on-Distance

$$\mathbf{q}_r = F + {}^{\star}\mathbf{q}_r$$
$$\frac{d\mathbf{q}_r}{dF} = 1$$
$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)$$
$$\therefore$$
$$\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big) = 0$$
$$\mathfrak{W}_{\bullet} = \mathfrak{P}$$

Time-on-Time

$$\mathbf{q}_r = E\,{}^{\star}\mathbf{q}_r$$
$$\frac{d\mathbf{q}_r}{dE} = {}^{\star}\mathbf{q}_r = \frac{\mathbf{q}_r}{E}$$
$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)\dfrac{\mathbf{q}_r}{E}$$
$$\therefore$$
$$\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)\mathbf{q}_r = 0$$
$$\mathfrak{W}_{\bullet\bullet} = \mathfrak{P}_{\bullet}$$

Time-on-Time-and-Distance

$$\mathbf{q}_r = F + E\,{}^{\star}\mathbf{q}_r$$
$$\frac{\partial\mathbf{q}_r}{\partial E} = {}^{\star}\mathbf{q}_r = \frac{\mathbf{q}_r - F}{E} \qquad \frac{\partial\mathbf{q}_r}{\partial F} = 1$$
$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)\dfrac{\mathbf{q}_r - F}{E}$$
$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)$$
$$\therefore$$
$$\sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)\mathbf{q}_r = 0 = \sum_r \mathbf{W}_r\big(\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)$$
$$\mathfrak{W}_{\bullet\bullet} = \mathfrak{P}_{\bullet} \qquad \text{and} \qquad \mathfrak{W}_{\bullet} = \mathfrak{P}$$

Where we have evaluated these fancy script *moments* at the general (transformed) $\mathbf{q}$

$$\mathfrak{W}_{\bullet} = \sum_r \mathbf{W}_r\,\mathbf{q}_r \qquad \mathfrak{P}_{\bullet} = \sum_r \overline{\overline{\mathbf{p}}}_r \mathbf{W}_r\,\mathbf{q}_r \qquad \mathfrak{W}_{\bullet\bullet} = \sum_r \mathbf{W}_r\,\mathbf{q}_r^2$$

We've translated our gauge criterion to be wholly applicable at the gauge transformed and optimal $\mathbf{q}$, very simply done by virtue of the linearity of the inverse gauge transformation. Recognizing that the relative gauge solution need also be a particular solution to the regression model makes even this simple bit of algebra redundant (i.e. we could have required $\mathbf{q} = {}^{\star}\mathbf{q}$ without loss of generality). But it is also possible to state the criterion by evaluating moments at the particular and non-optimal ${}^{\star}\mathbf{q}$ and explicitly solve for F, E or $\begin{bmatrix} E & F \end{bmatrix}$ so that the resulting transformed ${}^{\star}\mathbf{q} \mapsto \mathbf{q}$ meets the criterion

Time-on-Distance

$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(F + {}^{\star}\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)$$
$$\therefore$$
$$\mathfrak{W}F + \mathfrak{W}_{\star} = \mathfrak{P}$$
$$\therefore$$
$$F = \frac{\mathfrak{P} - \mathfrak{W}_{\star}}{\mathfrak{W}}$$
$${}^{\star}\mathbf{q}_r \mapsto F + {}^{\star}\mathbf{q}_r = \mathbf{q}_r$$
$${}^{\kappa}\mathbf{h}_{\star} \mapsto {}^{\kappa}\mathbf{h}_{\star} - F = {}^{\kappa}\mathbf{h}$$

Time-on-Time

$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(E\,{}^{\star}\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big){}^{\star}\mathbf{q}_r$$
$$\therefore$$
$$\mathfrak{W}_{\star\star}E = \mathfrak{P}_{\star}$$
$$\therefore$$
$$E = \frac{\mathfrak{P}_{\star}}{\mathfrak{W}_{\star\star}}$$
$${}^{\star}\mathbf{q}_r \mapsto E\,{}^{\star}\mathbf{q}_r = \mathbf{q}_r$$
$${}^{\kappa}\mathbf{k}_{\star} \mapsto \tfrac{{}^{\kappa}\mathbf{k}_{\star}}{E} = {}^{\kappa}\mathbf{k}$$

Time-on-Time-and-Distance

$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(F + E\,{}^{\star}\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big){}^{\star}\mathbf{q}_r$$
$$0 = 2\textstyle\sum_r \mathbf{W}_r\big(F + E\,{}^{\star}\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big)$$
$$\therefore$$
$$\begin{bmatrix} \mathfrak{W}_{\star\star} & \mathfrak{W}_{\star} \\ \mathfrak{W}_{\star} & \mathfrak{W} \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} \mathfrak{P}_{\star} \\ \mathfrak{P} \end{bmatrix}$$
$$\therefore$$
$$E = \frac{\mathfrak{P}_{\star}\mathfrak{W} - \mathfrak{P}\mathfrak{W}_{\star}}{\mathfrak{W}_{\star\star}\mathfrak{W} - \mathfrak{W}_{\star}^2} \text{ and } F = \frac{\mathfrak{W}_{\star\star}\mathfrak{P} - \mathfrak{W}_{\star}\mathfrak{P}_{\star}}{\mathfrak{W}_{\star\star}\mathfrak{W} - \mathfrak{W}_{\star}^2}$$
$${}^{\star}\mathbf{q}_r \mapsto F + E\,{}^{\star}\mathbf{q}_r = \mathbf{q}_r$$
$$\begin{bmatrix} {}^{\kappa}\mathbf{k}_{\star} & {}^{\kappa}\mathbf{h}_{\star} \end{bmatrix} \mapsto \begin{bmatrix} \frac{{}^{\kappa}\mathbf{k}_{\star}}{E} & \frac{E\,{}^{\kappa}\mathbf{k}_{\star} - F\,{}^{\kappa}\mathbf{h}_{\star}}{E} \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{k} & {}^{\kappa}\mathbf{h} \end{bmatrix}$$

Using starred moments in terms of the particular ${}^{\star}\mathbf{q}$ (non-optimal with respect to the gauge)

$$\mathfrak{W}_{\star} = \sum_r \mathbf{W}_r\,{}^{\star}\mathbf{q}_r \qquad \mathfrak{P}_{\star} = \sum_r \overline{\overline{\mathbf{p}}}_r \mathbf{W}_r\,{}^{\star}\mathbf{q}_r \qquad \mathfrak{W}_{\star\star} = \sum_r \mathbf{W}_r\,{}^{\star}\mathbf{q}_r^2$$

Here we annotate the moments with the same decoration as used for the specific solution. In general, these moments are dependent on the variable $\mathbf{q}$ and, as such, are themselves variables which require context to be fully defined. For the general $\mathbf{q}$ which satisfies the relative gauge criterion the context is not reflected in the notation for the moments. We can express the specific solution to the fleet relative gauge criterion as ${}^{\circledast}\mathbf{q}$ and the moments likewise using a $\circledast$ instead of a bullet $\bullet$ so that

Time-on-Distance

$$\mathfrak{W}_{\circledast} = \mathfrak{P}$$

Time-on-Time

$$\mathfrak{W}_{\circledast\circledast} = \mathfrak{P}_{\circledast}$$

Time-on-Time-and-Distance

$$\mathfrak{W}_{\circledast\circledast} = \mathfrak{P}_{\circledast} \qquad \text{and} \qquad \mathfrak{W}_{\circledast} = \mathfrak{P}$$

16.4.5 The Station Absolute Gauge as an Adjunct to the Fleet Relative Gauge

The specific solution ${}^{\circledast}\mathbf{q}$ to the regression model that also satisfies the fleet relative gauge criterion is a well defined as the fleet standard pace for a race. The specific solution ${}^{\boxast}\mathbf{q}$ for the station absolute gauge is then best defined through a gauge conversion from the fleet relative gauge

$$\text{fleet relative gauge} \qquad {}^{\circledast}\mathbf{q} \leftrightarrow {}^{\boxast}\mathbf{q} \qquad \text{station absolute gauge}$$

For single factor handicapping we need only make use of the station and fleet average pace $\overline{\overline{\overline{\mathbf{p}}}}$ defined such that

$$\mathfrak{W}\overline{\overline{\overline{\mathbf{p}}}} = \mathfrak{P}$$

For time-on-time-and-distance we will also need to make use of the fleet q-axis intercept ${}^{\circledast}\mathbf{q}_{\odot}$ for the relative gauge solution. Note that, by virtue of the relative gauge criteria, the $\overline{\overline{\overline{\mathbf{p}}}}$ and the ${}^{\circledast}\mathbf{q}_{\odot}$ will always be expressed in units of seconds per mile

Time-on-Distance

$$\begin{gathered} {}^{\circledast}\mathbf{q} \leftrightarrow {}^{\boxast}\mathbf{q} \\ \overline{\overline{\overline{\mathbf{p}}}} \mapsfrom 0\,\mathrm{s/mi} \\ \therefore \\ f = \overline{\overline{\overline{\mathbf{p}}}} \\ {}^{\circledast}\mathbf{q}_r \mapsto {}^{\circledast}\mathbf{q}_r - f = {}^{\boxast}\mathbf{q}_r \\ {}^{\kappa}\mathbf{h}_{\circledast} \mapsto {}^{\kappa}\mathbf{h}_{\circledast} + f = {}^{\kappa}\mathbf{h}_{\boxast} \end{gathered}$$

Time-on-Time

$$\begin{gathered} {}^{\circledast}\mathbf{q} \leftrightarrow {}^{\boxast}\mathbf{q} \\ \overline{\overline{\overline{\mathbf{p}}}} \mapsfrom 100\% \\ \therefore \\ e = \overline{\overline{\overline{\mathbf{p}}}} \\ {}^{\circledast}\mathbf{q}_r \mapsto \frac{{}^{\circledast}\mathbf{q}_r}{e} = {}^{\boxast}\mathbf{q}_r \\ {}^{\kappa}\mathbf{k}_{\circledast} \mapsto {}^{\kappa}\mathbf{k}_{\circledast} e = {}^{\kappa}\mathbf{k}_{\boxast} \end{gathered}$$

Time-on-Time-and-Distance

$$\begin{gathered} \begin{matrix} {}^{\circledast}\mathbf{q} \leftrightarrow {}^{\boxast}\mathbf{q} \\ \overline{\overline{\overline{\mathbf{p}}}} \mapsfrom 0\% \end{matrix} \quad \text{and} \quad \begin{matrix} {}^{\circledast}\mathbf{q} \leftrightarrow {}^{\boxast}\mathbf{q} \\ {}^{\circledast}\mathbf{q}_{\odot} \mapsfrom -100\% \end{matrix} \\ \therefore \\ \begin{bmatrix} e & f \end{bmatrix} = \begin{bmatrix} \overline{\overline{\overline{\mathbf{p}}}} - {}^{\circledast}\mathbf{q}_{\odot} & \overline{\overline{\overline{\mathbf{p}}}} \end{bmatrix} \\ {}^{\circledast}\mathbf{q} \mapsto \frac{{}^{\circledast}\mathbf{q} - f}{e} = {}^{\boxast}\mathbf{q}_r \\ \begin{bmatrix} {}^{\kappa}\mathbf{k}_{\circledast} & {}^{\kappa}\mathbf{h}_{\circledast} \end{bmatrix} \mapsto \begin{bmatrix} {}^{\kappa}\mathbf{k}_{\circledast} e & {}^{\kappa}\mathbf{h}_{\circledast} + {}^{\kappa}\mathbf{k}_{\circledast} f \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{k}_{\boxast} & {}^{\kappa}\mathbf{h}_{\boxast} \end{bmatrix} \end{gathered}$$

The fleet relative gauge solution for the nominal pace ${}^{\circledast}\mathbf{q}$ is in units of pace leading to handicaps in a relative gauge (with units as fits the gauge) whereas the station absolute gauge solution ${}^{\boxast}\mathbf{q}$ is unitless leading to handicaps in an absolute gauge.

A computed handicap for a particular boat determined with respect to this absolute gauge should not depend on the composition of the fleet to the same extent as would a relative gauge handicap. However by gaining a certain fleet independence we lose what independence we had from the expected paces suitable for the races which made up the data set. For single factor handicaps this wasn't much to start with but, for time-on-time-and-distance handicaps, this could be a significant loss of generality.

16.5 Time-on-Distance, Time-on-Time and the Log Transform

Suppose that we define the *log pace*

$${}^{\kappa}\mathfrak{p}_r^i = \log\left({}^{\kappa}\mathbf{p}_r^i\right) = \log\left({}^{\kappa}\mathbf{t}_r^i \div \mathbf{d}_r\right) = \log\left({}^{\kappa}\mathbf{t}_r^i\right) - \log\left(\mathbf{d}_r\right)$$

Then ${}^{\kappa}\overline{\mathfrak{p}}_r$ is either the arithmetic mean of the log pace or the log of the *geometric mean pace* ${}^{\kappa}\overline{\mathbf{p}}_r^{\mathrm{G}}$

$${}^{\kappa}\overline{\mathfrak{p}}_r = \tfrac{1}{{}^{\kappa}\mathbf{N}_r} \textstyle\sum_i {}^{\kappa}\overline{\mathfrak{p}}_r^i = \tfrac{1}{{}^{\kappa}\mathbf{N}_r} \textstyle\sum_i \log\left({}^{\kappa}\mathbf{p}_r^i\right) = \log\left(\sqrt[{}^{\kappa}\mathbf{N}_r]{\textstyle\prod_i {}^{\kappa}\mathbf{p}_r^i}\right) = \log\left({}^{\kappa}\overline{\mathbf{p}}_r^{\mathrm{G}}\right)$$

We then declare $\mathfrak{h}$ and $\mathfrak{q}$ as logarithmic counterparts to $\mathbf{h}$ and $\mathbf{q}$. Using the same computations as for time-on-distance handicapping we can derive ${}^{\kappa}\mathfrak{h}$ which can be transformed to time-on-time handicaps via an exponentiation ${}^{\kappa}\mathbf{k} = \exp\left({}^{\kappa}\mathfrak{h}\right)$. Note that the least squares minimization via the transformed $\mathfrak{p}$

yields different handicaps than would a direct least-squares minimization because of different handling of error terms

$$\sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left({}^\kappa\overline{\mathfrak{p}}_r - {}^\kappa\mathfrak{h} - \mathfrak{q}_r\right)^2 = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \Bigg[\underbrace{\log\left(\frac{{}^\kappa\overline{\mathbf{p}}_r^{\mathrm{G}}}{{}^\kappa\mathbf{k}\mathbf{q}_r}\right)}_{\text{logarithmic error}} \Bigg]^2 \quad \text{versus} \quad \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \big(\underbrace{{}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\mathbf{k}\mathbf{q}_r}_{\text{linear error}}\big)^2$$

The former is general linear model with a logarithmic error term and the latter a nonlinear model with a linear error term — statistically they are quite different. Historically time-on-time handicaps have been computed using the logarithmic model owing to the ease of computation rather than a strong belief in the error model — nevertheless a logarithmic model does seem more plausible than a linear one. From our perspective the only weakness of the logarithmic error model is that it makes it difficult to compare how well the derived time-on-time handicaps fit the data versus the general formulation which uses a linear error model.

The transformation makes it unnecessary to address the logarithmic time-on-time model directly and simply treat it as a by-product of the time-on-distance linear model.

Chapter 17

Time-on-Distance As a Linear Model

17.1 A Very Dense Presentation of Results

This chapter should be useful for those writing their own computer programme — others can skip this chapter and read the following chapter on using a linear model in the *R* statistics package.

General linear models are extremely well understood and thoroughly documented in textbooks and online. This summary in no way tries to explain the mathematics or prove assertions. It is paced for someone already very familiar with the material. Only the specific notation and contexts are made clear in order to state the results unambiguously.

17.2 Notation Within the Matrix Algebra

It is useful to consider our $\overline{\mathbf{p}}$, $\mathbf{P}$ and $\mathbf{N}$ arrays as rectangular matrices within the matrix algebra. Working backwards from the components ${}^{\kappa}\overline{\mathbf{p}}_r$, ${}^{\kappa}\mathbf{P}_r$ and ${}^{\kappa}\mathbf{N}_r$ we can refer to a column vector when we drop the superscript, a row vector when dropping a subscript, and a matrix when dropping both. So $\overline{\mathbf{p}}$, $\mathbf{P}$ and $\mathbf{N}$ will be matrices ranging over classes or boats down the rows and races across the columns, $\overline{\mathbf{p}}_r$ a column vector for race r, etc. This is the usual tabular layout for race results by boat and race but it does mean that the matrices suitable for handicapping purposes will typically have far more columns than rows.

The row vector $\mathbf{W} = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{N} = \sum_{\kappa} {}^{\kappa}\mathbf{N}$ is the a sum of the rows of $\mathbf{N}$ where $\mathbf{j}$ is the somewhat standard notation for a vector of all ones — more specifically $\mathbf{j}_{\mathfrak{K}}$ is a column of ones sized for the number of classes and $\mathbf{j}_{\mathfrak{R}}$ is a row of ones sized for the number of races. As a column vector, $\mathbf{M} = \mathbf{N}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} = \sum_r \mathbf{N}_r$. The zero column vector $\mathbf{0}_{\mathfrak{K}}$ and zero row vector $\mathbf{0}_{\mathfrak{R}}$ are sized like the corresponding $\mathbf{j}$ vector.

Standard square matrices will be sized for classes $\mathfrak{K}$ or races $\mathfrak{R}$: the identity matrices $\mathbf{I}_{\mathfrak{K}}$ or $\mathbf{I}_{\mathfrak{R}}$ with ones on the diagonal and zeros elsewhere and the matrices of all ones $\mathbf{J}_{\mathfrak{K}} = \mathbf{j}_{\mathfrak{K}}\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}$ or $\mathbf{J}_{\mathfrak{R}} = \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}\mathbf{j}_{\mathfrak{R}}$ respectively. The notation $\backslash\!\!\backslash\mathbf{u}\backslash\!\!\backslash$ will denote a square matrix with the components of a column or row vector $\mathbf{u}$ laid down the diagonal and zero elsewhere — this is the general diagonal matrix — multiplying a vector by a compatibly sized diagonal matrix just multiplies component-wise. For a row vector $\mathbf{u}$ of size n with components $\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}$ we can display the general diagonal matrix as

$$\backslash\!\!\backslash\mathbf{u}\backslash\!\!\backslash = \begin{bmatrix} \mathbf{u}_0 & 0 & \cdots & 0 \\ 0 & \mathbf{u}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{u}_{n-1} \end{bmatrix}$$

As specific cases $\mathbf{I}_{\mathfrak{K}} = \backslash\!\backslash \mathbf{j}_{\mathfrak{K}} \backslash\!\backslash$ and $\mathbf{I}_{\mathfrak{R}} = \backslash\!\backslash \mathbf{j}_{\mathfrak{R}} \backslash\!\backslash$ with the appropriate sizes.

17.3 The Stationarity Equations and Their Matrix Solution

For time-on-distance handicapping the Q and $\underline{Q}$ performance indices evaluate to quadratic forms with a linear solution

$$\underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\widehat{\mathbf{p}}_r\right]^2 = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\overline{\mathbf{p}}_r - \mathrm{cap}^\kappa(\mathbf{q}_r)\right]^2 = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\mathbf{h} - \mathbf{q}_r\right]^2$$

This minimization problem can be considered a standard weighted linear regression of a dependent variable with a grid of class×race samples onto class+race independent variables which simply indicate a corresponding column or row of the grid (to select the ${}^\kappa\mathbf{h}$ or $\mathbf{q}_r$ parameter respectively). This is quite elegant but, computationally, it is more straightforward to solve the stationarity equations directly. Dealing with the $\mathbf{q}^{\mathrm{T}}$ first as there are typically far more races than classes

$$0 = \frac{\partial \underline{Q}}{\partial \mathbf{q}_s} = \sum_\kappa 2\ {}^\kappa\mathbf{N}_s\left[{}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\mathbf{h} - \mathbf{q}_s\right](-1) \qquad 0 = \frac{\partial \underline{Q}}{\partial\, {}^\theta\mathbf{h}} = \sum_r 2\ {}^\theta\mathbf{N}_r\left[{}^\theta\overline{\mathbf{p}}_r - {}^\theta\mathbf{h} - \mathbf{q}_r\right](-1)$$

$$0 = \sum_\kappa {}^\kappa\mathbf{N}_r\left[\mathbf{q}_r + {}^\kappa\mathbf{h} - {}^\kappa\overline{\mathbf{p}}_r\right] \qquad 0 = \sum_r {}^\kappa\mathbf{N}_r\left[\mathbf{q}_r + {}^\kappa\mathbf{h} - {}^\kappa\overline{\mathbf{p}}_r\right]$$

Solving for $\mathbf{q}_r$ and ${}^\kappa\mathbf{h}$ respectively and noting $\mathbf{W} = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{N} = \sum_\kappa {}^\kappa\mathbf{N}$ and $\mathbf{M} = \mathbf{N}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} = \sum_r \mathbf{N}_r$

$$\mathbf{q}_r \sum_\kappa {}^\kappa\mathbf{N}_r + \sum_\kappa {}^\kappa\mathbf{N}_r\, {}^\kappa\mathbf{h} = \sum_\kappa {}^\kappa\mathbf{N}_r\, {}^\kappa\overline{\mathbf{p}}_r \qquad \sum_r {}^\kappa\mathbf{N}_r\mathbf{q}_r + {}^\kappa\mathbf{h}\sum_r {}^\kappa\mathbf{N}_r = \sum_r {}^\kappa\mathbf{N}_r\, {}^\kappa\overline{\mathbf{p}}_r$$

$$\mathbf{W}_r\, \mathbf{q}_r + \sum_\kappa {}^\kappa\mathbf{N}_r\, {}^\kappa\mathbf{h} = \sum_\kappa {}^\kappa\mathbf{P}_r \qquad \sum_r {}^\kappa\mathbf{N}_r\, \mathbf{q}_r + {}^\kappa\mathbf{M}\, {}^\kappa\mathbf{h} = \sum_r {}^\kappa\mathbf{P}_r$$

$$\backslash\!\backslash\mathbf{W}\backslash\!\backslash\mathbf{q}^{\mathrm{T}} + \mathbf{N}^{\mathrm{T}}\mathbf{h} = \mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \qquad \mathbf{N}\mathbf{q}^{\mathrm{T}} + \backslash\!\backslash\mathbf{M}\backslash\!\backslash\mathbf{h} = \mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}$$

As a partitioned matrix (with the numerous $\mathbf{q}_r$ before and over the less numerous ${}^\kappa\mathbf{h}$)

$$\begin{bmatrix} \backslash\!\backslash\mathbf{W}\backslash\!\backslash & \mathbf{N}^{\mathrm{T}} \\ \mathbf{N} & \backslash\!\backslash\mathbf{M}\backslash\!\backslash \end{bmatrix} \begin{bmatrix} \mathbf{q}^{\mathrm{T}} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \\ \mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \end{bmatrix}$$

This can be solved using block elimination on $\mathbf{q}^{\mathrm{T}}$ in the augmented matrix

$$\text{Schur complement} = \backslash\!\backslash\mathbf{M}\backslash\!\backslash - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}$$

$$\begin{bmatrix} \backslash\!\backslash\mathbf{W}\backslash\!\backslash & \mathbf{N}^{\mathrm{T}} & \mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \\ \mathbf{N} & \backslash\!\backslash\mathbf{M}\backslash\!\backslash & \mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \end{bmatrix} \overset{\text{row}}{\sim} \begin{bmatrix} \mathbf{I} & \backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} & \backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \\ \mathbf{0} & \backslash\!\backslash\mathbf{M}\backslash\!\backslash - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} & \mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash\mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \end{bmatrix}$$

Using this block reduced form and noting that $\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} = \backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{W}^{\mathrm{T}} = \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}$ we get the matrix equation on $\mathbf{h}$

$$\begin{aligned} \left(\backslash\!\backslash\mathbf{M}\backslash\!\backslash - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\right)\mathbf{h} &= \mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{K}} \\ &= \left(\mathbf{P}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}\right)\mathbf{j}_{\mathfrak{K}} \end{aligned}$$

This matrix equation is guaranteed to have many solutions and any two solutions can be made equivalent by applying gauge transformations within each league. Alternatively, we may eliminate on $\mathbf{h}$ to solve for $\mathbf{q}^{\mathrm{T}}$. The results are consistent between the two.

$$\begin{aligned} \left(\backslash\!\backslash\mathbf{W}\backslash\!\backslash - \mathbf{N}^{\mathrm{T}}\backslash\!\backslash\mathbf{M}\backslash\!\backslash^{-1}\mathbf{N}\right)\mathbf{q}^{\mathrm{T}} &= \mathbf{P}^{\mathrm{T}}\mathbf{j}_{\mathfrak{R}} - \mathbf{N}^{\mathrm{T}}\backslash\!\backslash\mathbf{M}\backslash\!\backslash^{-1}\mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \\ &= \left(\mathbf{P}^{\mathrm{T}}\backslash\!\backslash\mathbf{M}\backslash\!\backslash^{-1}\mathbf{N} - \mathbf{N}^{\mathrm{T}}\backslash\!\backslash\mathbf{M}\backslash\!\backslash^{-1}\mathbf{P}\right)\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \end{aligned}$$

17.4 The Relative and Absolute Gauge Criteria

Using the sum of the stationarity equations $\sum_s {}^{\partial Q}/_{\partial \mathbf{q}_s} = 0$ we can show that the fleet relative gauge stationarity criterion for time-on-distance $\mathbf{W}({}^{\circledast}\mathbf{q} - \overline{\overline{\mathbf{p}}})^{\mathrm{T}} = \sum_r \mathbf{W}_r\big({}^{\circledast}\mathbf{q}_r - \overline{\overline{\mathbf{p}}}_r\big) = 0$ is equivalent to the simpler condition

$$\mathbf{W}({}^{\circledast}\mathbf{q})^{\mathrm{T}} = \sum_r \mathbf{W}_r \cdot {}^{\circledast}q_r = \sum_r \mathbf{W}_r \overline{\overline{\mathbf{p}}}_r = \mathfrak{P} \qquad \Longleftrightarrow \qquad \mathbf{M}^{\mathrm{T}} \cdot \mathbf{h}_{\circledast} = \sum_\kappa {}^{\kappa}\mathbf{M} \cdot {}^{\kappa}\mathbf{h}_{\circledast} = 0$$

The absolute gauge criterion is specified by the condition

$$\mathbf{W}({}^{\boxast}\mathbf{q})^{\mathrm{T}} = \sum_r \mathbf{W} \cdot {}^{\boxast}\mathbf{q}_r = 0 \qquad \Longleftrightarrow \qquad \mathbf{M}^{\mathrm{T}} \cdot \mathbf{h}_{\boxast} = \sum_\kappa {}^{\kappa}\mathbf{M} \cdot {}^{\kappa}\mathbf{h}_{\boxast} = \mathfrak{P}$$

17.5 Two-Way Analysis of Variance

It should be noted that the least-squares estimation of ${}^{\kappa}\widehat{\mathbf{p}}_r$ in

$$\sum_r \sum_\kappa \sum_i \big[{}^{\kappa}\mathbf{p}_r^i - {}^{\kappa}\widehat{\mathbf{p}}_r\big]^2 = \sum_r \sum_\kappa \sum_i \big[{}^{\kappa}\mathbf{p}_r^i - \mathrm{cap}^{\kappa}(\mathbf{q}_r)\big]^2 = \sum_r \sum_\kappa \sum_i \big[{}^{\kappa}\mathbf{p}_r^i - {}^{\kappa}\mathbf{h} - \mathbf{q}_r\big]^2$$

is an instance of a standard two-way analysis-of-variance problem complicated only by the different number of finishers in each race. All the standard statistical methods still apply. Nevertheless, we will use the specific notation defined above to identify sums-of-squares rather than using a generic notation.

$$\begin{array}{rcll} \mathrm{SS_E} & = & \underline{\mathrm{SS}}_\mathrm{E} & +\mathrm{SS_W} \\ +\mathrm{SS_R} & & +\mathrm{SS_R} & \\ \hline \mathrm{SS_A} & = & \underline{\mathrm{SS}}_\mathrm{A} & +\mathrm{SS_W} \end{array} \qquad \begin{array}{rcll} \nu_\mathrm{E} & = & \underline{\nu}_\mathrm{E} & +\nu_\mathrm{W} \\ +\nu_\mathrm{R} & & +\nu_\mathrm{R} & \\ \hline \nu_\mathrm{A} & = & \underline{\nu}_\mathrm{A} & +\nu_\mathrm{W} \end{array}$$

Source of Variation	Sum of Squares	°Freedom	Mean Square	Root Mean Square
Pure Error	$\mathrm{SS_W}$	$\nu_\mathrm{W} = \nu_p - \nu_{\bar{p}}$	$\mathrm{MS_W} = \frac{\mathrm{SS_W}}{\nu_\mathrm{W}}$	$\mathrm{RMS_W} = \sqrt{\mathrm{MS_W}}$
Lack of Fit	$\mathrm{SS_B} = \underline{\mathrm{SS}}_\mathrm{E}$	$\nu_\mathrm{B} = \nu_{\bar{p}} - \nu_{\hat{p}}$	$\mathrm{MS_B} = \frac{\mathrm{SS_B}}{\nu_\mathrm{B}}$	
Residual Error	$\mathrm{SS_E} = \underline{\mathrm{SS}}_\mathrm{E} + \mathrm{SS_W}$	$\nu_\mathrm{E} = \nu_p - \nu_{\hat{p}}$	$\mathrm{MS_E} = \frac{\mathrm{SS_E}}{\nu_\mathrm{E}}$	$\mathrm{RMS_E} = \sqrt{\mathrm{MS_E}}$
Regression Model	$\mathrm{SS_R}$	$\nu_\mathrm{R} = \nu_{\hat{p}} - \nu_{\bar{\bar{p}}}$	$\mathrm{MS_R} = \frac{\mathrm{SS_R}}{\nu_\mathrm{R}}$	
All Told	$\mathrm{SS_A} = \underline{\mathrm{SS}}_\mathrm{A} + \mathrm{SS_W}$	$\nu_\mathrm{A} = \nu_p - \nu_{\bar{\bar{p}}}$	$\mathrm{MS_A} = \frac{\mathrm{SS_A}}{\nu_\mathrm{A}}$	$\mathrm{RMS_A} = \sqrt{\mathrm{MS_A}}$

$\mathrm{RMS_A}$ estimates the deviation in pace between all boats in a race, $\mathrm{RMS_E}$ the deviation in corrected pace and $\mathrm{RMS_W}$ the deviation in pace between boats within each class. The regression model coefficient of determination R^2 (and also its positive square root, the multiple correlation coefficient R) is defined

$$R^2 = \frac{\mathrm{SS_R}}{\mathrm{SS_A}} = \frac{\underline{\mathrm{SS}}_\mathrm{A} - \underline{\mathrm{SS}}_\mathrm{E}}{\mathrm{SS_W} + \underline{\mathrm{SS}}_\mathrm{A}}$$

This measures how much of the variability in the finishing paces of boats can be ascribed to the handicapping formula for classes. We can partition the residual error to determine the lack of fit coefficient of determination R^2_B and the F_B statistic as defined

$$R^2_\mathrm{B} = \frac{\mathrm{SS_B}}{\mathrm{SS_E}} = \frac{\underline{\mathrm{SS}}_\mathrm{E}}{\mathrm{SS_W} + \underline{\mathrm{SS}}_\mathrm{E}} \qquad F_\mathrm{B} = \frac{\mathrm{MS_B}}{\mathrm{MS_W}} = \frac{\mathrm{SS_B} \div \nu_\mathrm{B}}{\mathrm{SS_W} \div \nu_\mathrm{W}} = \frac{\underline{\mathrm{SS}}_\mathrm{E}}{\mathrm{SS_W}} \div \frac{\nu_{\bar{p}} - \nu_{\hat{p}}}{\nu_p - \nu_{\bar{p}}}$$

The F_{B} statistic is used to test for goodness of fit under the assumption of a normally-distributed error model and can be used to look up an easily interpreted "*p-value*" statistic via a $F(\nu_{\mathrm{B}}, \nu_{\mathrm{W}})$ tail distribution function (i.e. for a Snedecor F random variable with $\nu_{\mathrm{B}} = \underline{\nu}_{\mathrm{E}} = \nu_{\bar{p}} - \nu_{\hat{p}}$ numerator degrees of freedom and $\nu_{\mathrm{W}} = \nu_p - \nu_{\bar{p}}$ denominator degrees of freedom).

The F_{B} statistic and the $p_{F_{\mathrm{B}}}$ statistic derived from it are used to test the null hypothesis that the regressed handicaps are no better than level-racing. Here $p_{F_{\mathrm{B}}}$ is a *p-value* statistic and unrelated to pace — the overuse of the letter "p" to represent both concepts is an unfortunate coincidence. If instead of applying our regression model to observed paces ${}^\kappa\mathbf{p}^i_r$ we apply them to corrected pace with existing but possibly non-optimal handicaps we can compare the newly regressed handicaps to the existing handicaps. This yields a F_{B} statistic and a $p_{F_{\mathrm{B}}}$ statistic which tests the null hypothesis that the regressed handicaps are no better the existing handicaps. Were the *p-value* $p_{F_{\mathrm{B}}} < 5\%$ we would reject the null hypothesis and say the regressed handicaps are *significantly* better than the existing handicaps. That is, given the variability in pace between boats as observed in the given data set, the odds are less than 5% that the regressed handicaps achieve their better predictive power by a mere chance alignment of numbers. On the other hand, one time in twenty, an existing perfect set of handicaps would fail this test at random. Were we to repeat this test often we would chose a lower threshold that 5%.

Also the F_{B} test cannot account for systematic *errors* in the data set such as local sailing conditions, winds, currents and course configurations which favour one class over another or a difference in the experience and abilities of the crew. Unless you set up an experimental design to isolate these factors they will inevitably be conflated in the regressed handicaps. And whether you consider these as *errors* or as a proper subject for performance handicapping cannot meaningfully be addressed by one statistical test.

17.6 The Linear Model in a Progressive Sense

The terms of the performance index for the general linear model can be written in a progressive sense that is equivalent to the regressive one

$$\begin{array}{rcccccccc}
{}^\kappa\check{\mathbf{q}}^i_r - \mathbf{q}_r & \equiv & \mathrm{chk}^\kappa({}^\kappa\mathbf{p}^i_r) - \mathbf{q}_r & = & {}^\kappa\mathbf{p}^i_r - {}^\kappa\mathbf{h} - \mathbf{q}_r & = & {}^\kappa\mathbf{p}^i_r - \mathrm{cap}^\kappa(\mathbf{q}_r) & = & {}^\kappa\mathbf{p}^i_r - {}^\kappa\widehat{\mathbf{p}}_r \\
{}^\kappa\check{\overline{\mathbf{q}}}_r - \mathbf{q}_r & \equiv & \mathrm{chk}^\kappa({}^\kappa\overline{\mathbf{p}}_r) - \mathbf{q}_r & = & {}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\mathbf{h} - \mathbf{q}_r & = & {}^\kappa\overline{\mathbf{p}}_r - \mathrm{cap}^\kappa(\mathbf{q}_r) & = & {}^\kappa\overline{\mathbf{p}}_r - {}^\kappa\widehat{\mathbf{p}}_r
\end{array}$$

$$Q = \sum_r \sum_\kappa \sum_i \left[{}^\kappa\check{\mathbf{q}}^i_r - \mathbf{q}_r\right]^2 \qquad\qquad \underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\check{\overline{\mathbf{q}}}_r - \mathbf{q}_r\right]^2$$

17.6.1 Explicitly Relative Forms for Handicapping Operations

We wish to infer results from the relative performance of boats, and therefore take particular interest in an alternative formulation that only depends on explicitly relative terms. Handicaps shall be computed to minimize the simplest reasonable quadratic performance index: within a given race r we take the sum of squares of the difference in handicapped pace $\left|{}^\kappa\check{\overline{\mathbf{q}}}_r - {}^\lambda\check{\overline{\mathbf{q}}}_r\right|$ over all combinations $\{\kappa, \lambda\}$ of any two classes $\binom{\mathfrak{K}}{2}$; we account for the multiplicity for all such pair of boats in the race ${}^\kappa\mathbf{N}_r \times {}^\lambda\mathbf{N}_r$ noting that when either ${}^\kappa\mathbf{N}_r$ or ${}^\lambda\mathbf{N}_r$ is zero the whole term is eliminated from the sum; we balance the oversampling of boats within the race by dividing by a factor of $\frac{1}{\mathbf{W}_r}$ where $\mathbf{W}_r$ is the total number who participated in this race; then we sum over all races. We will call expressions of this form a boat-by-boat sum-of-squares. Of the $|\mathfrak{K}|$ total number of superscripts there are $\left|\binom{\mathfrak{K}}{2}\right| = \frac{|\mathfrak{K}| \times (|\mathfrak{K}|-1)}{2}$ paired combinations. Algebraically, it is simpler to analyze the equivalent sum of squares with κ and

λ ranging independently over $\mathfrak{K}$ — when $\kappa = \lambda$ all the terms are zero — when $\kappa \neq \lambda$ the $(\kappa,\ \lambda)$ term equals the $(\lambda,\ \kappa)$ term — and then divide by 2 to compensate for this oversampling

$$^{\mathrm{BB}}\underline{Q} = \sum_r \frac{1}{\mathbf{W}_r} \sum\sum_{\{\kappa,\lambda\}\in\binom{\mathfrak{K}}{2}} {}^{\kappa}\mathbf{N}_r\,{}^{\lambda}\mathbf{N}_r \left|{}^{\kappa}\check{\overline{\mathbf{q}}}_r - {}^{\lambda}\check{\overline{\mathbf{q}}}_r\right|^2 = \sum_r \frac{1}{2\mathbf{W}_r} \sum_{\kappa\in\mathfrak{K}} {}^{\kappa}\mathbf{N}_r \sum_{\lambda\in\mathfrak{K}} {}^{\lambda}\mathbf{N}_r \left[{}^{\kappa}\check{\overline{\mathbf{q}}}_r - {}^{\lambda}\check{\overline{\mathbf{q}}}_r\right]^2$$

The stationarity equations on the $^{\mathrm{BB}}\underline{Q}$ can be written as a familiar matrix equation

$$\left(\backslash\!\backslash\mathbf{M}\backslash\!\backslash - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\right)\mathbf{h} = \left(\mathbf{P}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}\right)\mathbf{j}_{\mathfrak{K}}$$

The above is boat-by-boat (actually class-by-class) sum of squares in the reduced model. There is a full model boat-by-boat sum-of-squares

$$^{\mathrm{BB}}Q = \mathrm{SS}_{\mathrm{W}} + {}^{\mathrm{BB}}\underline{Q}$$

For a given race r, the i spans the boats of class κ and the j spans the boats of class λ respectively

$$^{\mathrm{BB}}Q = \sum_r \frac{1}{\mathbf{W}_r} \sum\sum_{\{\kappa,\lambda\}\in\binom{\mathfrak{K}}{2}} \sum_i \sum_j \left|{}^{\kappa}\check{\mathbf{q}}_r^i - {}^{\lambda}\check{\mathbf{q}}_r^j\right|^2 = \sum_r \frac{1}{2\mathbf{W}_r} \sum_{\kappa\in\mathfrak{K}} \sum_i \sum_{\lambda\in\mathfrak{K}} \sum_j \left[{}^{\kappa}\check{\mathbf{q}}_r^i - {}^{\lambda}\check{\mathbf{q}}_r^j\right]^2$$

This has a simple interpretation. Let $\mathfrak{G}_r$ be the individual boats that participated in race r, $\binom{\mathfrak{G}_r}{2}$ all combinations $\{\alpha,\ \beta\}$ of any two boats in the race and $\left|\check{\mathbf{q}}_r^{\alpha} - \check{\mathbf{q}}_r^{\beta}\right|$ the difference in handicapped pace between any two individual boats in the race so that, in exactly the same terms as above,

$$^{\mathrm{BB}}Q = \sum_r \frac{1}{\mathbf{W}_r} \sum\sum_{\{\alpha,\beta\}\in\binom{\mathfrak{G}_r}{2}} \left|\check{\mathbf{q}}_r^{\alpha} - \check{\mathbf{q}}_r^{\beta}\right|^2 = \sum_r \frac{1}{2\mathbf{W}_r} \sum_{\alpha\in\mathfrak{G}_r} \sum_{\beta\in\mathfrak{G}_r} \left[\check{\mathbf{q}}_r^{\alpha} - \check{\mathbf{q}}_r^{\beta}\right]^2$$

17.6.2 Achieving Optimal Performance via a Boat-by-Boat Pairwise Model

The optimal $\mathbf{q}_r$ can be easily expressed in terms of the optimal ${}^{\kappa}\check{\overline{\mathbf{q}}}_r = {}^{\kappa}\overline{\mathbf{p}}_r - {}^{\kappa}\mathbf{h}$ from the stationarity equations for the standard regressive model

$$q_r = \frac{1}{\mathbf{W}_r} \sum_{\kappa} {}^{\kappa}\mathbf{N}_r\,{}^{\kappa}\check{\overline{\mathbf{q}}}_r$$

Independently of the stationarity equations, using only the techniques of completing and partitioning a square, we can show for arbitrary $\mathbf{q}_r$ and ${}^{\kappa}\mathbf{h}$ the performance indices $^{\mathrm{BB}}\underline{Q}$ and $\underline{Q}$ are related

$$\underline{Q} = {}^{\mathrm{BB}}\underline{Q} + \sum_r \mathbf{W}_r \left[\mathbf{q}_r - \frac{1}{\mathbf{W}_r} \sum_{\kappa} {}^{\kappa}\mathbf{N}_r\,{}^{\kappa}\check{\overline{\mathbf{q}}}_r\right]^2$$

At the optimal $\mathbf{q}_r$ and ${}^{\kappa}\mathbf{h}$ the sums of squares are equal

$$\sum_r \sum_{\kappa} {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\check{\overline{\mathbf{q}}}_r - \mathbf{q}_r\right]^2 = \underline{\mathrm{SS}}_{\mathrm{E}} = \sum_r \frac{1}{2\mathbf{W}_r} \sum_{\alpha} {}^{\alpha}\mathbf{N}_r \sum_{\beta} {}^{\beta}\mathbf{N}_r \left[{}^{\alpha}\check{\overline{\mathbf{q}}}_r - {}^{\beta}\check{\overline{\mathbf{q}}}_r\right]^2$$

As are the all told sums of squares

$$\sum_r \sum_{\kappa} {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\overline{\mathbf{p}}_r - \overline{\overline{\mathbf{p}}}_r\right]^2 = \underline{\mathrm{SS}}_{\mathrm{A}} = \sum_r \frac{1}{2\mathbf{W}_r} \sum_{\alpha} {}^{\alpha}\mathbf{N}_r \sum_{\beta} {}^{\beta}\mathbf{N}_r \left[{}^{\alpha}\overline{\mathbf{p}}_r - {}^{\beta}\overline{\mathbf{p}}_r\right]^2$$

17.6.3 A Surprising Duality

It turns out that the general linear model is equivalent, not only to the boat-by-boat pairwise model, but also to the race-by-race pairwise model. All three models minimize their performance index to the same optimal value $\underline{\mathrm{SS}}_{\mathrm{E}}$ and at the corresponding values of their $\mathbf{h}$ or $\mathbf{q}$ free parameters.

$$
\begin{aligned}
{}^{\mathrm{BB}}\underline{Q} &= \sum_r \frac{1}{2\mathbf{W}_r} \sum_\kappa {}^{\kappa}\mathbf{N}_r \sum_\lambda {}^{\lambda}\mathbf{N}_r \big[({}^{\kappa}\overline{\mathbf{p}}_r - {}^{\kappa}\mathbf{h}) - ({}^{\lambda}\overline{\mathbf{p}}_r - {}^{\lambda}\mathbf{h})\big]^2 \\
\underline{Q} &= \sum_r \sum_\kappa {}^{\kappa}\mathbf{N}_r \big[{}^{\kappa}\overline{\mathbf{p}}_r - {}^{\kappa}\mathbf{h} - \mathbf{q}_r\big]^2 \\
{}^{\mathrm{RR}}\underline{Q} &= \sum_\kappa \frac{1}{2\,{}^{\kappa}\mathbf{M}} \sum_r {}^{\kappa}\mathbf{N}_r \sum_s {}^{\kappa}\mathbf{N}_s \big[({}^{\kappa}\overline{\mathbf{p}}_r - \mathbf{q}_r) - ({}^{\kappa}\overline{\mathbf{p}}_s - \mathbf{q}_s)\big]^2
\end{aligned}
$$

In general, for any ${}^{\kappa}n_r$, ${}^{\kappa}y_r$, q_r and ${}^{\kappa}h$ (note the shorthand ${}^{\backslash}n_r = \sum_\kappa {}^{\kappa}n_r$ and ${}^{\kappa}n_{\backslash} = \sum_r {}^{\kappa}n_r$)

$$
\begin{aligned}
&\sum_r \sum_\kappa {}^{\kappa}n_r \big[{}^{\kappa}y_r - {}^{\kappa}h - q_r\big]^2 \\
&= \sum_r \frac{1}{{}^{\backslash}n_r} \left\{ \frac{1}{2} \sum_\kappa {}^{\kappa}n_r \sum_\lambda {}^{\lambda}n_r \big[({}^{\kappa}y_r - {}^{\kappa}h) - ({}^{\lambda}y_r - {}^{\lambda}h)\big]^2 \;+\; \left[\sum_\kappa {}^{\kappa}n_r({}^{\kappa}y_r - {}^{\kappa}h - q_r)\right]^2 \right\} \\
&= \sum_\kappa \frac{1}{{}^{\kappa}n_{\backslash}} \left\{ \frac{1}{2} \sum_r {}^{\kappa}n_r \sum_s {}^{\kappa}n_s \big[({}^{\kappa}y_r - q_r) - ({}^{\kappa}y_s - q_s)\big]^2 \;+\; \left[\sum_r {}^{\kappa}n_r({}^{\kappa}y_r - {}^{\kappa}h - q_r)\right]^2 \right\}
\end{aligned}
$$

These algebraic identities, while unexpected, aren't difficult to verify. They encapsulate everything needed to know in order to equate the three models.

17.7 The Competition Matrices

17.7.1 Diagonally-Dominated Nonnegative-Definite Symmetric Forms X (Chi)

Tensor X, the Competition Matrices $\mathbf{X}_r$ and their Weighted Average $\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r = \mathbf{\Omega}$ From the number of boats in each class $\mathbf{N}_r$ for a given fixed race r we define a square symmetric matrix $\mathbf{X}_r$ with positive entries on the diagonal and with negative entries off the diagonal

$$
{}^{\alpha\alpha}\mathbf{X}_r = \sum_{\omega\neq\alpha} {}^{\alpha}\mathbf{N}_r\, {}^{\omega}\mathbf{N}_r = {}^{\alpha}\mathbf{N}_r \sum_{\omega\neq\alpha} {}^{\omega}\mathbf{N}_r = {}^{\alpha}\mathbf{N}_r\big(\mathbf{W}_r - {}^{\alpha}\mathbf{N}_r\big) \qquad {}^{\alpha\beta}\mathbf{X}_r = -\,{}^{\alpha}\mathbf{N}_r\, {}^{\beta}\mathbf{N}_r \text{ when } \alpha \neq \beta
$$

This matrix can be written in terms of the outer square of the $\mathbf{N}_r$ vector and the all-ones $\mathbf{j}_{\mathfrak{K}}$ column vector

$$
\mathbf{X}_r = \backslash\!\!\backslash \mathbf{N}_r \mathbf{W}_r \backslash\!\!\backslash - \mathbf{N}_r \mathbf{N}_r^{\mathrm{T}} = \backslash\!\!\backslash \mathbf{N}_r \mathbf{N}_r^{\mathrm{T}} \mathbf{j}_{\mathfrak{K}} \backslash\!\!\backslash - \mathbf{N}_r \mathbf{N}_r^{\mathrm{T}}
$$

Clearly $\mathbf{X}_r$ satisfies the equation $\mathbf{X}_r \mathbf{j}_{\mathfrak{K}} = \mathbf{0}$. By definition it also just satisfies the criterion of diagonal dominance (the inequality necessary to satisfy the dominance criterion is met by an actual equality)

$$
|{}^{\beta\beta}\mathbf{X}_r| = \sum_{\omega\neq\beta} |{}^{\omega\beta}\mathbf{X}_r|
$$

On one hand, we can consider $\mathbf{X}_r$ a linear transformation mapping $\mathbf{u}$ to $\mathbf{v}$

$$
\mathbf{v} = \mathbf{X}_r \mathbf{u} \quad \Longleftrightarrow \quad \text{for all } \alpha \Rightarrow {}^{\alpha}\mathbf{v} = \sum_\beta {}^{\alpha\beta}\mathbf{X}_r\, {}^{\beta}\mathbf{u} = \sum_\beta {}^{\alpha}\mathbf{N}_r\, {}^{\beta}\mathbf{N}_r \left({}^{\alpha}\mathbf{u} - {}^{\beta}\mathbf{u}\right)
$$

Or, on the other hand, we can consider $\mathbf{X}_r$ a symmetric bilinear form acting on $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u}^{\mathrm{T}}\mathbf{X}_r\mathbf{v} = \sum_\alpha\sum_\beta {}^{\alpha\beta}\mathbf{X}_r\,{}^{\alpha}\mathbf{u}\,{}^{\beta}\mathbf{v} = \frac{1}{2}\sum_\alpha\sum_\beta({}^{\alpha}\mathbf{u}-{}^{\beta}\mathbf{u}){}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{N}_r({}^{\alpha}\mathbf{v}-{}^{\beta}\mathbf{v})$$

Which, by repeating the $\mathbf{u}$, gives us a nonnegative quadratic form

$$\mathbf{u}^{\mathrm{T}}\mathbf{X}_r\mathbf{u} = \frac{1}{2}\sum_\alpha\sum_\beta {}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{N}_r({}^{\alpha}\mathbf{u}-{}^{\beta}\mathbf{u})^2 \geq 0$$

These properties all also hold for the linear combination $\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r = \backslash\!\!\backslash\mathbf{M}\backslash\!\!\backslash - \mathbf{N}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} = \boldsymbol{\Omega}$

$$\mathbf{u}^{\mathrm{T}}\Big(\sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\Big)\mathbf{v} = \sum_\alpha\sum_\beta\Big(\sum_r\frac{1}{\mathbf{W}_r}\,{}^{\alpha\beta}\mathbf{X}_r\Big)\,{}^{\alpha}\mathbf{u}\,{}^{\beta}\mathbf{v} = \frac{1}{2}\sum_\alpha\sum_\beta({}^{\alpha}\mathbf{u}-{}^{\beta}\mathbf{u})\Big(\sum_r\frac{{}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{N}_r}{\mathbf{W}_r}\Big)({}^{\alpha}\mathbf{v}-{}^{\beta}\mathbf{v})$$

$$\mathbf{u}^{\mathrm{T}}\Big(\sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\Big)\mathbf{u} = \frac{1}{2}\sum_\alpha\sum_\beta\Big(\sum_r\frac{{}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{N}_r}{\mathbf{W}_r}\Big)({}^{\alpha}\mathbf{u}-{}^{\beta}\mathbf{u})^2 \geq 0$$

The $\mathbf{X}_r$ and $\sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r$ are diagonally-dominated nonnegative-definite symmetric matrices. The pairwise performance index ${}^{\mathrm{BB}}\underline{Q}$ can be written in terms of the $\mathbf{X}_r$ and $\check{\overline{\mathbf{q}}}_r = \overline{\mathbf{p}}_r - \mathbf{h}$ as can its gradient with respect to the vector $\mathbf{h}$

$${}^{\mathrm{BB}}\underline{Q} = \sum_r\frac{1}{\mathbf{W}_r}\check{\overline{\mathbf{q}}}_r^{\mathrm{T}}\mathbf{X}_r\check{\overline{\mathbf{q}}}_r \qquad\qquad \nabla_{\mathbf{h}}{}^{\mathrm{BB}}\underline{Q} = -\sum_r\frac{2}{\mathbf{W}_r}\mathbf{X}_r\check{\overline{\mathbf{q}}}_r$$

Which yields another interesting variation on the solution to the pairwise stationarity equation

$$\nabla_{\mathbf{h}}{}^{\mathrm{BB}}\underline{Q} = \mathbf{0} \qquad\Longrightarrow\qquad \sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\check{\overline{\mathbf{q}}}_r = \mathbf{0} \qquad\Longrightarrow\qquad \Big(\sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\Big)\mathbf{h} = \sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\overline{\mathbf{p}}_r$$

And formulae for the various sums-of-squares at the optimal $\mathbf{h}$

$$\underline{\mathrm{SS}}_{\mathrm{A}} = \sum_r\frac{1}{\mathbf{W}_r}\overline{\mathbf{p}}_r^{\mathrm{T}}\mathbf{X}_r\overline{\mathbf{p}}_r \qquad \mathrm{SS}_{\mathrm{R}} = \mathbf{h}^{\mathrm{T}}\Big(\sum_r\frac{1}{\mathbf{W}_r}\mathbf{X}_r\Big)\mathbf{h} \qquad \underline{\mathrm{SS}}_{\mathrm{E}} = \sum_r\frac{1}{\mathbf{W}_r}(\overline{\mathbf{p}}_r-\mathbf{h})^{\mathrm{T}}\mathbf{X}_r(\overline{\mathbf{p}}_r-\mathbf{h})$$

And by stationarity we have a very direct and easy inference that $\underline{\mathrm{SS}}_{\mathrm{A}} = \mathrm{SS}_{\mathrm{R}} + \underline{\mathrm{SS}}_{\mathrm{E}}$.

17.7.2 Antisymmetric Forms Y (Upsilon)

Tensor Y, the Competition Matrices $\mathbf{Y}_r$ and their Weighted Average $\sum_r\frac{1}{\mathbf{W}_r}\mathbf{Y}_r = \boldsymbol{\Psi}$ From the $\mathbf{N}_r$ and $\overline{\mathbf{p}}_r$ for a given fixed race r we define a square skew-symmetric matrix $\mathbf{Y}_r$

$${}^{\alpha\beta}\mathbf{Y}_r = {}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{N}_r\left({}^{\alpha}\overline{\mathbf{p}}_r - {}^{\beta}\overline{\mathbf{p}}_r\right) = {}^{\alpha}\mathbf{P}_r\,{}^{\beta}\mathbf{N}_r - {}^{\alpha}\mathbf{N}_r\,{}^{\beta}\mathbf{P}_r$$

This matrix can be written in terms of the outer square of the $\mathbf{N}_r$ vector and the diagonal $\backslash\!\!\backslash\overline{\mathbf{p}}_r\backslash\!\!\backslash$

$$\mathbf{Y}_r = \backslash\!\!\backslash\overline{\mathbf{p}}_r\backslash\!\!\backslash\mathbf{N}_r\mathbf{N}_r^{\mathrm{T}} - \mathbf{N}_r\mathbf{N}_r^{\mathrm{T}}\backslash\!\!\backslash\overline{\mathbf{p}}_r\backslash\!\!\backslash = \mathbf{P}_r\mathbf{N}_r^{\mathrm{T}} - \mathbf{N}_r\mathbf{P}_r^{\mathrm{T}}$$

$$\sum_r\frac{1}{\mathbf{W}_r}\mathbf{Y}_r = \mathbf{P}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} - \mathbf{N}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}$$

We can consider $\mathbf{Y}_r$ and $\sum_r\frac{1}{\mathbf{W}_r}\mathbf{Y}_r$ as alternating bilinear forms $\mathbf{u}^{\mathrm{T}}\mathbf{Y}_r\mathbf{u} = 0 = \mathbf{u}^{\mathrm{T}}\left(\sum_r\frac{1}{\mathbf{W}_r}\mathbf{Y}_r\right)\mathbf{u}$

$$\begin{aligned}&\mathbf{u}^{\mathrm{T}}\mathbf{Y}_r\mathbf{v}\\ &= \mathbf{u}^{\mathrm{T}}\mathbf{P}_r\mathbf{N}_r^{\mathrm{T}}\mathbf{v} - \mathbf{v}^{\mathrm{T}}\mathbf{P}_r\mathbf{N}_r^{\mathrm{T}}\mathbf{u}\\ &= \left(\mathbf{P}_r^{\mathrm{T}}\mathbf{u}\right)^{\mathrm{T}}\left(\mathbf{N}_r^{\mathrm{T}}\mathbf{v}\right) - \left(\mathbf{P}_r^{\mathrm{T}}\mathbf{v}\right)^{\mathrm{T}}\left(\mathbf{N}_r^{\mathrm{T}}\mathbf{u}\right)\end{aligned} \qquad \begin{aligned}&\mathbf{u}^{\mathrm{T}}\left(\textstyle\sum_r\frac{1}{\mathbf{W}_r}\mathbf{Y}_r\right)\mathbf{v}\\ &= \mathbf{u}^{\mathrm{T}}\mathbf{P}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{v} - \mathbf{v}^{\mathrm{T}}\mathbf{P}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{u}\\ &= \left(\mathbf{P}^{\mathrm{T}}\mathbf{u}\right)^{\mathrm{T}}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\left(\mathbf{N}^{\mathrm{T}}\mathbf{v}\right) - \left(\mathbf{P}^{\mathrm{T}}\mathbf{v}\right)^{\mathrm{T}}\backslash\!\!\backslash\mathbf{W}\backslash\!\!\backslash^{-1}\left(\mathbf{N}^{\mathrm{T}}\mathbf{u}\right)\end{aligned}$$

17.7.3 The Combined Symmetric Forms X and Antisymmetric Forms Y

When applied to $\mathbf{j}_{\mathfrak{K}}$ the $\mathbf{Y}_r$ and the $\sum_r \frac{1}{\mathbf{W}_r}\mathbf{Y}_r$ give results applicable to the stationarity equation

$$\sum_\beta {}^{\alpha\beta}\mathbf{Y}_r = \sum_\beta {}^{\alpha}\mathbf{N}_r\, {}^{\beta}\mathbf{N}_r \left({}^{\alpha}\overline{\mathbf{p}}_r - {}^{\beta}\overline{\mathbf{p}}_r\right) = \sum_\beta {}^{\alpha\beta}\mathbf{X}_r\, {}^{\beta}\overline{\mathbf{p}}_r$$

$$\begin{array}{ll} \mathbf{Y}_r\mathbf{j}_{\mathfrak{K}} = \mathbf{X}_r\mathbf{p}_r & \left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{Y}_r\right)\mathbf{j}_{\mathfrak{K}} = \left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r\overline{\mathbf{p}}_r\right) = \left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r\right)\mathbf{h} \\ 0 = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{Y}_r\mathbf{j}_{\mathfrak{K}} = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{X}_r\overline{\mathbf{p}}_r & 0 = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{Y}_r\right)\mathbf{j}_{\mathfrak{K}} = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r\overline{\mathbf{p}}_r\right) = \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\left(\sum_r \frac{1}{\mathbf{W}_r}\mathbf{X}_r\right)\mathbf{h} \end{array}$$

17.8 Point Solutions to the Matrix Equations

17.8.1 Where the Handicaps Sum to Zero

Lets write the stationarity equation

$$\underbrace{\left(\backslash\!\backslash\mathbf{M}\backslash\!\backslash - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}}\right)}_{\mathbf{\Omega}}\mathbf{h} = \underbrace{\left(\mathbf{P}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{N}^{\mathrm{T}} - \mathbf{N}\backslash\!\backslash\mathbf{W}\backslash\!\backslash^{-1}\mathbf{P}^{\mathrm{T}}\right)}_{\mathbf{\Psi}}\mathbf{j}_{\mathfrak{K}}$$

where $\mathbf{\Omega}$ is a diagonally-dominant nonnegative-definite symmetric such that $\mathbf{\Omega}\mathbf{j}_{\mathfrak{K}} = \mathbf{0}$ and $\mathbf{\Psi}$ is skew-symmetric. The minimization problem ensures that the matrix equation can be solved for $\mathbf{h}$ and that solution space has a single dimension for each league. It is convenient to restrict our attention to within a single league of $|\mathfrak{K}|$ classes where the solution space is one-dimensional $\mathbf{h}_0 + \langle\mathbf{j}_{\mathfrak{K}}\rangle$ and where $\mathbf{h}_0$ is the particular solution such that all the handicaps sum to zero. This can be computed directly by left multiplying $\mathbf{\Psi}\mathbf{j}_{\mathfrak{K}}$ by the Moore-Penrose generalized inverse of $\mathbf{\Omega}$.

We can decompose the space of handicaps $\mathbb{R}^{|\mathfrak{K}|} = \langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp} \oplus \langle\mathbf{j}_{\mathfrak{K}}\rangle$ as a direct sum of the particular solution space $\langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp} = \{\mathbf{h}_0 \mid \mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{h}_0 = 0\}$ (all handicaps summing to zero) and the space of gauge transformations $\langle\mathbf{j}_{\mathfrak{K}}\rangle$ (constant offsets from the zero-summing solutions). The Moore-Penrose generalized inverse acts on $\langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp} \oplus \langle\mathbf{j}_{\mathfrak{K}}\rangle$ mapping the $\langle\mathbf{j}_{\mathfrak{K}}\rangle$ space to zero and solving for $\mathbf{h}$ wholly within $\langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp}$. Complementary to the $\mathbf{\Omega}$ is the matrix of all ones $\mathbf{J}_{\mathfrak{K}}$ which acts wholly within $\langle\mathbf{j}_{\mathfrak{K}}\rangle$ while mapping $\langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp}$ to zero. Together these allow for a nicely computable point solution $\mathbf{h}_0 = \mathbf{\Omega}_\omega^{-1}\mathbf{\Psi}\mathbf{j}_{\mathfrak{K}}$ by means of the symmetric matrix $\mathbf{\Omega}_\omega = \mathbf{\Omega} + \frac{\omega}{|\mathfrak{K}|}\mathbf{J}_{\mathfrak{K}}$ which will be invertible provided $\omega \neq 0$, nonnegative-definite when $\omega \geq 0$ and positive-definite when $\omega > 0$. The two halves of the $\mathbf{\Omega}_\omega$ act independently within $\langle\mathbf{j}_{\mathfrak{K}}\rangle^{\perp}$ and $\langle\mathbf{j}_{\mathfrak{K}}\rangle$ respectively.

17.8.2 Shifting the Gauge from where the Handicaps Sum to Zero

Making use of this direct sum decomposition we can, for any γ, solve directly into the chosen gauge where the handicaps $\mathbf{h}_\gamma = \mathbf{h}_0 + \mathbf{j}_{\mathfrak{K}}\frac{\gamma}{|\mathfrak{K}|}$ sum to the target $\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{h}_\gamma = \gamma$ rather than zero

$$\left(\mathbf{\Omega} + \frac{\omega}{|\mathfrak{K}|}\mathbf{J}_{\mathfrak{K}}\right)\left(\mathbf{h}_0 + \mathbf{j}_{\mathfrak{K}}\frac{\gamma}{|\mathfrak{K}|}\right) = \mathbf{\Psi}\mathbf{j}_{\mathfrak{K}} + \mathbf{j}_{\mathfrak{K}}\frac{\gamma\omega}{|\mathfrak{K}|} \quad\Longrightarrow\quad \mathbf{h}_\gamma = \mathbf{\Omega}_\omega^{-1}\left(\mathbf{\Psi}\mathbf{j}_{\mathfrak{K}} + \mathbf{j}_{\mathfrak{K}}\frac{\gamma\omega}{|\mathfrak{K}|}\right)$$

But this isn't as useful as it would at first appear — the sum of handicaps isn't a particularly meaningful measure and it can not be reasonably targeted — it is unrelated to either the relative gauge or the absolute gauge criterion. Calculating any particular solution $\mathbf{h}_\star$ first and then solving for F in order to balance the final result for the relative gauge

$$\mathbf{M}^{\mathrm{T}}(\mathbf{h}_\star - \mathbf{j}_{\mathfrak{K}}F) = 0 \quad\Longrightarrow\quad F = \frac{\mathbf{M}^{\mathrm{T}}\cdot\mathbf{h}_\star}{\mathbf{M}^{\mathrm{T}}\cdot\mathbf{j}_{\mathfrak{K}}}$$

yields a more significant solution

$$\mathbf{h}_{\circledast} = \mathbf{h}_{\star} - \mathbf{j}_{\mathfrak{K}} \frac{\mathbf{M}^{\mathrm{T}} \cdot \mathbf{h}_{\star}}{\mathbf{M}^{\mathrm{T}} \cdot \mathbf{j}_{\mathfrak{K}}} \quad \text{such that} \quad \begin{cases} \boldsymbol{\Omega} \mathbf{h}_{\circledast} & = \boldsymbol{\Psi} \mathbf{j}_{\mathfrak{K}} \\ \mathbf{M}^{\mathrm{T}} \cdot \mathbf{h}_{\circledast} & = 0 \end{cases}$$

Similarly we can solve for $\mathbf{q}^{\mathrm{T}}$

$$\big(\underbrace{\backslash\!\!\backslash \mathbf{W} \backslash\!\!\backslash - \mathbf{N}^{\mathrm{T}} \backslash\!\!\backslash \mathbf{M} \backslash\!\!\backslash^{-1} \mathbf{N}}_{\mathbf{S}} \big) \mathbf{q}^{\mathrm{T}} = \big(\underbrace{\mathbf{P}^{\mathrm{T}} \backslash\!\!\backslash \mathbf{M} \backslash\!\!\backslash^{-1} \mathbf{N} - \mathbf{N}^{\mathrm{T}} \backslash\!\!\backslash \mathbf{M} \backslash\!\!\backslash^{-1} \mathbf{P}}_{\mathbf{A}} \big) \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}$$

$${}^{0}\mathbf{q}^{\mathrm{T}} = \big(\mathbf{S} + \tfrac{s}{|\mathfrak{R}|} \mathbf{J}_{\mathfrak{R}} \big)^{-1} \mathbf{A} \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \;\text{ and then }\; {}^{\boxast}\mathbf{q}^{\mathrm{T}} = {}^{0}\mathbf{q}^{\mathrm{T}} - \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \frac{\mathbf{W} \cdot {}^{0}\mathbf{q}^{\mathrm{T}}}{\mathbf{W} \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}} \;\text{ solving }\; \begin{cases} \mathbf{S} \cdot {}^{\boxast}\mathbf{q}^{\mathrm{T}} & = \mathbf{A} \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \\ \mathbf{W} \cdot {}^{\boxast}\mathbf{q}^{\mathrm{T}} & = 0 \end{cases}$$

17.8.3 Forcing the Gauge

Using the point solutions available within a single league, we can solve directly into the relative or absolute gauges through a variation on the stationarity equations. The ${}^{\kappa}\mathring{\mathbf{m}}$ and $\mathring{\mathbf{w}}_r$ are explicit secondary weights independent of the implicit primary weights ${}^{\kappa}\mathbf{m}$ and $\mathbf{w}_r$ already encompassed by the ${}^{\kappa}\mathbf{N}_r$ and ${}^{\kappa}\mathbf{P}_r$

$$\begin{aligned} {}^{\kappa}\mathring{\mathbf{m}} &= \sqrt{{}^{\kappa}\mathbf{M}} & \mathring{\mathbf{m}} &= \backslash\!\!\backslash \mathbf{M} \backslash\!\!\backslash^{1/2} \mathbf{j}_{\mathfrak{K}} & \mathring{\mathbf{m}} \mathring{\mathbf{m}}^{\mathrm{T}} &= \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash \mathbf{J}_{\mathfrak{K}} \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash \\ \mathring{\mathbf{w}}_r &= \sqrt{\mathbf{W}_r} & \mathring{\mathbf{w}} &= \mathbf{j}_{\mathfrak{R}} \backslash\!\!\backslash \mathbf{W} \backslash\!\!\backslash^{1/2} & \mathring{\mathbf{w}}^{\mathrm{T}} \mathring{\mathbf{w}} &= \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash \mathbf{J}_{\mathfrak{R}} \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash \end{aligned}$$

$${}^{\kappa}\mathring{\mathbf{N}}_r = \frac{{}^{\kappa}\mathbf{N}_r}{\sqrt{{}^{\kappa}\mathbf{M} \mathbf{W}_r}} \qquad {}^{\kappa}\mathring{\mathbf{P}}_r = \frac{{}^{\kappa}\mathbf{P}_r}{\sqrt{{}^{\kappa}\mathbf{M} \mathbf{W}_r}} \qquad \begin{aligned} \mathring{\mathbf{N}} &= \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash^{-1} \mathbf{N} \, \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash^{-1} \\ \mathring{\mathbf{P}} &= \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash^{-1} \mathbf{P} \, \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash^{-1} \end{aligned}$$

Here we decompose $\mathbb{R}^{|\mathfrak{K}|} = \langle \mathring{\mathbf{m}} \rangle^{\perp} \oplus \langle \mathring{\mathbf{m}} \rangle$ for the direct solution $\mathbf{h}_{\circledast}$ in the relative gauge

$$\mathbf{h}_{\circledast} = \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash^{-1} \big(\mathbf{I}_{\mathfrak{K}} - \mathring{\mathbf{N}} \mathring{\mathbf{N}}^{\mathrm{T}} + \mathring{\mathbf{m}} \tfrac{\omega}{\mathfrak{W}} \mathring{\mathbf{m}}^{\mathrm{T}} \big)^{-1} \big(\mathring{\mathbf{P}} \mathring{\mathbf{N}}^{\mathrm{T}} - \mathring{\mathbf{N}} \mathring{\mathbf{P}}^{\mathrm{T}} \big) \backslash\!\!\backslash \mathring{\mathbf{m}} \backslash\!\!\backslash \, \mathbf{j}_{\mathfrak{K}}$$

And we decompose $\mathbb{R}^{|\mathfrak{R}|} = \langle \mathring{\mathbf{w}} \rangle^{\perp} \oplus \langle \mathring{\mathbf{w}} \rangle$ for the direct solution ${}^{\boxast}\mathbf{q}^{\mathrm{T}}$ in a absolute gauge

$${}^{\boxast}\mathbf{q}^{\mathrm{T}} = \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash^{-1} \big(\mathbf{I}_{\mathfrak{R}} - \mathring{\mathbf{N}}^{\mathrm{T}} \mathring{\mathbf{N}} + \mathring{\mathbf{w}}^{\mathrm{T}} \tfrac{s}{\mathfrak{W}} \mathring{\mathbf{w}} \big)^{-1} \big(\mathring{\mathbf{P}}^{\mathrm{T}} \mathring{\mathbf{N}} - \mathring{\mathbf{N}}^{\mathrm{T}} \mathring{\mathbf{P}} \big) \backslash\!\!\backslash \mathring{\mathbf{w}} \backslash\!\!\backslash \, \mathbf{j}_{\mathfrak{R}}^{\mathrm{T}}$$

The $\mathfrak{W} = \sum_r \mathbf{W}_r = \mathring{\mathbf{w}} \mathring{\mathbf{w}}^{\mathrm{T}} = \mathring{\mathbf{m}}^{\mathrm{T}} \mathring{\mathbf{m}} = \sum_{\kappa} {}^{\kappa}\mathbf{M}$ is a convenient scaling factor.

Chapter 18

Solvers for Nonlinear Models

18.1 Least Squares for Time-on-Time and Time-on-Time-and-Distance

18.1.1 Multivariate Polynomials versus Simple Solvers

It's convenient to tweak the standard layout of the performance index $\underline{Q}$

$$\underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\widehat{\mathbf{p}}_r - {}^\kappa\overline{\mathbf{p}}_r\right]^2 = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[\mathrm{cap}^\kappa(\mathbf{q}_r) - {}^\kappa\overline{\mathbf{p}}_r\right]^2$$

$$\underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\mathbf{k} + \mathbf{q}_r - {}^\kappa\overline{\mathbf{p}}_r\right]^2 \qquad \text{(time-on-distance for comparison purposes)}$$

$$\underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\mathbf{k} \cdot \mathbf{q}_r - {}^\kappa\overline{\mathbf{p}}_r\right]^2 \qquad \text{(time-on-time)}$$

$$\underline{Q} = \sum_r \sum_\kappa {}^\kappa\mathbf{N}_r \left[{}^\kappa\mathbf{h} + {}^\kappa\mathbf{k} \cdot \mathbf{q}_r - {}^\kappa\overline{\mathbf{p}}_r\right]^2 \qquad \text{(time-on-time-and-distance)}$$

For time-on-time and time-on-time-and-distance handicapping the $\underline{Q}$ performance indices evaluate to multivariate polynomials in $\mathbf{q}_r$, ${}^\kappa\mathbf{k}$ and ${}^\kappa\mathbf{h}$. The minimization problems lacks an analytic solution and must be solved numerically. For efficiency and ease of analysis, solvers always restrict their inputs to a single league.

Iterative solvers use an initial guess for the control (the $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{h}$), test it against the criteria for optimality, and then refine the guess using the local first and second derivatives to estimate the behaviour of the whole. The more local information used to refine the estimate the better the guess — but we have to weigh the quality of the guess against the resources needed to compute it. The expected time to completion combines the expected number of iterations necessary to achieve the desired level of precision with the time to complete a single iteration. A solver with a very simple but quickly computed iterative step may lead to a solution in less time than a more sophisticated solver using fewer iterations.

18.1.2 Moment Variables in Terms of the Reduced Model

In computing a solution to the stationarity equations it is useful to define some moment variables that integrate the current state of the $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{h}$ with the fixed parameters $\mathbf{N}$ and $\overline{\mathbf{p}}$. First we will declare some matrices by componentwise multiplication, using a $\bullet$ or a $\circ$ to annotate the $\mathbf{N}$ or the $\mathbf{P}$. The *moments* simply sum over the resulting components, replacing an index by a backslash to sum over all possible indices. By distributing the multiplication appropriately we can write the doubly summed moments quite nicely in terms of the ${}^{\backslash}\mathbf{N}_r = \mathbf{W}_r$, the all told average $\overline{\overline{\mathbf{p}}}_r$ and the $\mathbf{q}_r$

$$
\begin{array}{ccccc|l}
\mathbf{N} & \mathbf{P} & \mathbf{N}_{\bullet}{=}\,\mathbf{N}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash & \mathbf{P}_{\bullet}{=}\mathbf{P}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash & \mathbf{N}_{\bullet\bullet}{=}\mathbf{N}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash^2 & \\
{}^{\bullet}\mathbf{N}{=}\,\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash\mathbf{N} & {}^{\bullet}\mathbf{P}{=}\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash\mathbf{P} & {}^{\bullet}\mathbf{N}_{\bullet}{=}\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash\mathbf{N}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash & & & \\
{}^{\circ}\mathbf{N}{=}\,\backslash\!\!\backslash\mathbf{h}\backslash\!\!\backslash\mathbf{N} & & {}^{\circ}\mathbf{N}_{\bullet}{=}\backslash\!\!\backslash\mathbf{h}\backslash\!\!\backslash\mathbf{N}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash & & {}^{\kappa}\mathbf{k}\cdot{}^{\kappa}\mathbf{h}\cdot{}^{\kappa}\mathbf{N}_r{=}{}^{\kappa\bullet\circ}\mathbf{N}_r & \backslash\bullet\circ \\
{}^{\bullet\bullet}\mathbf{N}{=}\,\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash^2\mathbf{N} & & & & {}^{\kappa}\mathbf{k}^2\cdot{}^{\kappa}\mathbf{N}_r{=}{}^{\kappa\bullet\bullet}\mathbf{N}_r & \backslash\bullet\bullet \\
{}^{\bullet\circ}\mathbf{N}{=}\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash\backslash\!\!\backslash\mathbf{h}\backslash\!\!\backslash\mathbf{N} & & {}^{\kappa}\mathbf{h}\cdot{}^{\kappa}\mathbf{N}_r\cdot\mathbf{q}_r{=}{}^{\kappa\circ}\mathbf{N}_{\bullet r} & & {}^{\kappa}\mathbf{h}\cdot{}^{\kappa}\mathbf{N}_r{=}\;{}^{\kappa\circ}\mathbf{N}_r & \backslash\circ \\
& & {}^{\kappa}\mathbf{k}\cdot{}^{\kappa}\mathbf{N}_r\cdot\mathbf{q}_r{=}{}^{\kappa\bullet}\mathbf{N}_{\bullet r} & {}^{\kappa}\mathbf{k}\cdot{}^{\kappa}\mathbf{N}_r\cdot{}^{\kappa}\overline{\mathbf{p}}_r{=}{}^{\kappa\bullet}\mathbf{P}_r & {}^{\kappa}\mathbf{k}\cdot{}^{\kappa}\mathbf{N}_r{=}\;{}^{\kappa\bullet}\mathbf{N}_r & \backslash\bullet \\
{}^{\kappa}\mathbf{N}_r\cdot\mathbf{q}_r^2{=}{}^{\kappa}\mathbf{N}_{\bullet\bullet r} & {}^{\kappa}\mathbf{N}_r\cdot{}^{\kappa}\overline{\mathbf{p}}_r\cdot\mathbf{q}_r{=}{}^{\kappa}\mathbf{P}_{\bullet r} & {}^{\kappa}\mathbf{N}_r\cdot\mathbf{q}_r{=}\;{}^{\kappa}\mathbf{N}_{\bullet r} & {}^{\kappa}\mathbf{N}_r\cdot{}^{\kappa}\overline{\mathbf{p}}_r{=}\;{}^{\kappa}\mathbf{P}_r & {}^{\kappa}\mathbf{N}_r & \backslash \\
\hline
\mathbf{N}_{\bullet\bullet\backslash} & \mathbf{P}_{\bullet\backslash} & \mathbf{N}_{\bullet\backslash} & \mathbf{P}_{\backslash} & \mathbf{N}_{\backslash} & \\
\sum_r{}^{\backslash}\mathbf{N}_r\mathbf{q}_r^2{=}{}^{\backslash}\mathbf{N}_{\bullet\bullet\backslash} & \sum_r{}^{\backslash}\mathbf{N}_r\overline{\overline{\mathbf{p}}}_r\mathbf{q}_r{=}{}^{\backslash}\mathbf{P}_{\bullet\backslash} & \sum_r{}^{\backslash}\mathbf{N}_r\mathbf{q}_r{=}{}^{\backslash}\mathbf{N}_{\bullet\backslash} & \sum_r{}^{\backslash}\mathbf{N}_r\overline{\overline{\mathbf{p}}}_r{=}{}^{\backslash}\mathbf{P}_{\backslash} & \sum_r{}^{\backslash}\mathbf{N}_r{=}{}^{\backslash}\mathbf{N}_{\backslash} &
\end{array}
$$

Declaring *moment vectors* directly can be nicer than referring to their componentwise counterparts

$$
\begin{array}{c}
{}^{\backslash}\mathbf{N}{=}\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{N} \qquad {}^{\backslash}\mathbf{P}{=}\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}\mathbf{P} \qquad {}^{\backslash\bullet}\mathbf{N}{=}\mathbf{k}^{\mathrm{T}}\mathbf{N} \qquad {}^{\backslash\bullet}\mathbf{P}{=}\mathbf{k}^{\mathrm{T}}\mathbf{P} \qquad {}^{\backslash\bullet\bullet}\mathbf{N}{=}\mathbf{k}^{\mathrm{T}}\backslash\!\!\backslash\mathbf{k}\backslash\!\!\backslash\mathbf{N} \qquad {}^{\backslash\bullet\circ}\mathbf{N}{=}\mathbf{k}^{\mathrm{T}}\backslash\!\!\backslash\mathbf{h}\backslash\!\!\backslash\mathbf{N} \\
\mathbf{N}_{\backslash}{=}\mathbf{N}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \qquad \mathbf{P}_{\backslash}{=}\mathbf{P}\mathbf{j}_{\mathfrak{R}}^{\mathrm{T}} \qquad \mathbf{N}_{\bullet\backslash}{=}\mathbf{N}\mathbf{q}^{\mathrm{T}} \qquad \mathbf{P}_{\bullet\backslash}{=}\mathbf{P}\mathbf{q}^{\mathrm{T}} \qquad \mathbf{N}_{\bullet\bullet\backslash}{=}\mathbf{N}\backslash\!\!\backslash\mathbf{q}\backslash\!\!\backslash\mathbf{q}^{\mathrm{T}}
\end{array}
$$

Of the moments which are synonymous with earlier declarations we can see they had been named so as to mimic their appearance in this notation

$$
\mathbf{W} = {}^{\backslash}\mathbf{N} \qquad \mathbf{M} = \mathbf{N}_{\backslash} \qquad \mathfrak{W} = {}^{\backslash}\mathbf{N}_{\backslash} \qquad \mathfrak{P} = {}^{\backslash}\mathbf{P}_{\backslash} \qquad \mathfrak{W}_{\bullet} = {}^{\backslash}\mathbf{N}_{\bullet\backslash} \qquad \mathfrak{P}_{\bullet} = {}^{\backslash}\mathbf{P}_{\bullet\backslash} \qquad \mathfrak{W}_{\bullet\bullet} = {}^{\backslash}\mathbf{N}_{\bullet\bullet\backslash}
$$

The ${}^{\backslash\bullet}\mathbf{N}$, $\mathbf{N}_{\bullet\backslash}$, ${}^{\backslash\bullet}\mathbf{P}$ and $\mathbf{P}_{\bullet\backslash}$ are first-order moments that vary from from control point to control point. These are akin to moments seen in centre-of-mass calculations where the *mass* (the $\mathbf{N}$ or $\mathbf{P}$) is distributed across *space* (the $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{h}$ components of state which designate *position* in the control space). In this sense the *masses* are fixed but their *position* changes according to the state. The ${}^{\backslash\bullet\bullet}\mathbf{N}$ and $\mathbf{N}_{\bullet\bullet\backslash}$ are second-order moments akin to moments of inertia. The bare ${}^{\backslash}\mathbf{N}$, $\mathbf{N}_{\backslash}$, ${}^{\backslash}\mathbf{P}$, $\mathbf{P}_{\backslash}$ and are zeroth-order moment vectors independent of the controls. The ${}^{\circ}\mathbf{N}$, ${}^{\bullet}\mathbf{N}$ and ${}^{\bullet}\mathbf{N}_{\bullet}$ aren't moments, as such, but it is useful to have them defined in this context.

The *second-factor discriminant* ${}^{\kappa}\mathbf{N}_{\triangle}$ is defined for each κ in $\mathfrak{K}$ in terms of second, first and zeroth-order moments

$$
{}^{\kappa}\mathbf{N}_{\triangle} = \begin{vmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\backslash} \end{vmatrix} = \begin{vmatrix} \mathbf{q}\backslash\!\!\backslash{}^{\kappa}\mathbf{N}\backslash\!\!\backslash\mathbf{q}^{\mathrm{T}} & \mathbf{q}\backslash\!\!\backslash{}^{\kappa}\mathbf{N}\backslash\!\!\backslash\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}} \\ \mathbf{j}_{\mathfrak{K}}\backslash\!\!\backslash{}^{\kappa}\mathbf{N}\backslash\!\!\backslash\mathbf{q}^{\mathrm{T}} & \mathbf{j}_{\mathfrak{K}}\backslash\!\!\backslash{}^{\kappa}\mathbf{N}\backslash\!\!\backslash\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}} \end{vmatrix} \geq 0
$$

Then as vectors

$$
\backslash\!\!\backslash\mathbf{N}_{\triangle}\backslash\!\!\backslash = \backslash\!\!\backslash\mathbf{N}_{\bullet\bullet\backslash}\backslash\!\!\backslash\backslash\!\!\backslash\mathbf{N}_{\backslash}\backslash\!\!\backslash - \backslash\!\!\backslash\mathbf{N}_{\bullet\backslash}\backslash\!\!\backslash^2
$$

The Schwartz inequality ensures ${}^{\kappa}\mathbf{N}_{\triangle}$ is nonnegative. It also informs us when the ${}^{\kappa}\mathbf{N}_{\triangle}$ must be zero. Consider all the $\mathbf{q}_r$ for which ${}^{\kappa}\mathbf{N}_r > 0$. The ${}^{\kappa}\mathbf{N}_{\triangle}$ will be zero whenever all such $\mathbf{q}_r$ are equal to each other — i.e. when $\mathbf{q}^{\mathrm{T}}$ and $\mathbf{j}_{\mathfrak{K}}^{\mathrm{T}}$ are parallel with respect to the $(\bullet\backslash\!\!\backslash{}^{\kappa}\mathbf{N}\backslash\!\!\backslash\bullet)$ inner product. When, for a given $\mathbf{q}$, the second-factor discriminant ${}^{\kappa}\mathbf{N}_{\triangle}$ is zero then the corresponding two-factor handicap will be underdetermined. A very small discriminant will lead to handicaps with poor predictive power away from the conditions in which the handicap was determined. Not surprisingly, having raced in a variety of races is sufficient for a large ${}^{\kappa}\mathbf{N}_{\triangle}$ and for each of the two factors of a handicap to be independently well specified.

In the context of an iterative solver the zeroth order moments are simply fixed parameters but the first and second order moments are *variables* as they are implicitly dependent on the control. These moments arise when we look at the first and second order derivatives of the cost $\underline{Q}$ around a given control point (the $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{h}$). For each iteration of the solver these moments will need to be recalculated with the most current estimate of the optimal control point.

18.2 Ping-Pong Iterative Solver for Time-on-Time

This is a first-order solver that ping-pongs back and forth between optimizing for all ${}^{\kappa}\mathbf{k}$ (at a given $\mathbf{q}$) and then for all $\mathbf{q}_r$ (at a given $\mathbf{k}$). It is so easy to code and each iteration is so fast that it hardly seems worthwhile investigating more sophisticated second-order methods.

18.2.1 The Algorithm

Staring with an initial guess for $\mathbf{q}$ and $\mathbf{k}$ each iteration of the ping-pong algorithm will come up with a lower value for the performance index $\underline{Q}$ which converge on the final $\underline{\mathrm{SS}}_{\mathrm{E}}$

$$\underline{Q} = \sum_r \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]^2$$

$${}^{\backslash\bullet\bullet}\mathbf{N}_r\,\mathbf{q}_r = {}^{\backslash\bullet}\mathbf{P}_r \quad \Longleftrightarrow \quad \begin{cases} \text{for each } r \\ \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] {}^{\kappa}\mathbf{k} = 0 \end{cases}$$

$${}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}\,{}^{\kappa}\mathbf{k} = {}^{\kappa}\mathbf{P}_{\bullet\backslash} \quad \Longleftrightarrow \quad \begin{cases} \text{for each } \kappa \\ \sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] \mathbf{q}_r = 0 \end{cases}$$

The algorithm stops once the difference between successive terms of the $\underline{Q}$ becomes sufficiently small. This is a first-order algorithm with slow but predictable convergence properties. It will usually take many more iterations than a second-order solver like Gauss's method but because each iteration of this algorithm is so simple it can complete many many iterations before a second order solver has completed just one.

With an initial value either $\mathbf{q} = \overline{\overline{\mathbf{p}}}$ for the relative gauge or $\mathbf{q} = \mathbf{j}_{\mathfrak{R}}$ for the absolute gauge

Repeat

> **Ping** discard the previous $\mathbf{k}$
>
> **For each** κ calculate a replacement value using the constant $\mathbf{N}$, $\overline{\overline{\mathbf{p}}}$ and variable $\mathbf{q}$
>
> $$0 = \frac{\partial \underline{Q}}{\partial\, {}^{\kappa}\mathbf{k}} \qquad \Longrightarrow \qquad {}^{\kappa}\mathbf{k} = \frac{{}^{\kappa}\mathbf{P}_{\bullet\backslash}}{{}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}}$$
>
> This yields the vector $\mathbf{k}$ which minimizes the performance index $\underline{Q}$ at the current value of $\mathbf{q}$ which remains fixed in this step of the algorithm. At this value of $\mathbf{q}$ there is no more optimization which can be done so we must improve the individual $\mathbf{q}_r$ to continue.
>
> **Pong** discard the previous $\mathbf{q}$
>
> **For each** r calculate a replacement value using the constant $\mathbf{N}$, $\overline{\mathbf{p}}$ and variable $\mathbf{k}$
>
> $$0 = \frac{\partial \underline{Q}}{\partial \mathbf{q}_r} \qquad \Longrightarrow \qquad \mathbf{q}_r = \frac{{}^{\backslash\bullet}\mathbf{P}_r}{{}^{\backslash\bullet\bullet}\mathbf{N}_r}$$
>
> This yields the vector $\mathbf{q}$ which minimizes the performance index $\underline{Q}$ at the current value of $\mathbf{k}$ which remains fixed in this step of the algorithm. At this value of $\mathbf{k}$ there is no more optimization which can be done so we must improve the individual ${}^{\kappa}\mathbf{k}$ to continue.

Continue while cost $\underline{Q}$ converges to a minimum.

18.2.2 On Stability of the Solutions

The algorithm throws away old values for the $\mathbf{q}$ and $\mathbf{k}$ as it goes along so you shouldn't expect the values for $\mathbf{q}$ and $\mathbf{k}$ to stabilize. Nevertheless, the iterative steps mostly preserve the gauge criterion chosen for the initial value and, as this is the only freedom allowed for the solutions, they do indeed converge as the algorithm proceeds. The gauge restriction will slowly drift unless nudged while the algorithm progresses; although, there is no particular reason to do so; enforcing the gauge criterion at the finish of the algorithm is more than sufficient.

18.3 Ping-Pong Iterative Solver for Time-on-Time-and-Distance

18.3.1 The Algorithm for an Arbitrary Data Set (Not Necessarily Seeded)

Staring with an initial guess for $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{h}$ each iteration of the ping-pong algorithm will come up with a lower value for the performance index $\underline{Q}$ which converge on the final $\underline{\underline{\mathrm{SS}}}_{\mathrm{E}}$

$$\underline{Q} = \sum_r \sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]^2$$

for each r

$$\sum_\kappa {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] {}^{\kappa}\mathbf{k} = 0 \qquad\qquad {}^{\backslash\bullet\bullet}\mathbf{N}_r\mathbf{q}_r + {}^{\backslash\bullet\circ}\mathbf{N}_r = {}^{\backslash\bullet}\mathbf{P}_r$$

for each κ

$$\sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] \mathbf{q}_r = 0$$

$$\sum_r {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k}\mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right] = 0 \qquad \begin{bmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\backslash} \end{bmatrix} \begin{bmatrix} {}^{\kappa}\mathbf{k} \\ {}^{\kappa}\mathbf{h} \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{P}_{\backslash} \end{bmatrix}$$

Just as for time-on-time, this algorithm has slow but predictable convergence properties. It will usually complete very quickly. For good numerical stability we will only ever start the algorithm in the relative gauge.

With an initial value $\mathbf{q} = \overline{\overline{\mathbf{p}}}$

Repeat

Ping discard the previous $\mathbf{k}$ and $\mathbf{h}$

For each κ calculate replacement values using the constant $\mathbf{N}$, $\overline{\mathbf{p}}$ and variable $\mathbf{q}$

$$\begin{bmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\backslash} \end{bmatrix} \begin{bmatrix} {}^{\kappa}\mathbf{k} \\ {}^{\kappa}\mathbf{h} \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{P}_{\backslash} \end{bmatrix} \qquad \text{whenever} \qquad {}^{\kappa}\mathbf{N}_{\Delta} = \begin{vmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\backslash} \end{vmatrix} > 0$$

for solutions by Cramer's rule

$${}^{\kappa}\mathbf{k} = \begin{vmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{P}_{\backslash} & {}^{\kappa}\mathbf{N}_{\backslash} \end{vmatrix} \frac{1}{{}^{\kappa}\mathbf{N}_{\Delta}} = \frac{{}^{\kappa}\mathbf{P}_{\bullet\backslash} \cdot {}^{\kappa}\mathbf{N}_{\backslash} - {}^{\kappa}\mathbf{P}_{\backslash} \cdot {}^{\kappa}\mathbf{N}_{\bullet\backslash}}{{}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} \cdot {}^{\kappa}\mathbf{N}_{\backslash} - ({}^{\kappa}\mathbf{N}_{\bullet\backslash})^2}$$

$${}^{\kappa}\mathbf{h} = \begin{vmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash} & {}^{\kappa}\mathbf{P}_{\backslash} \end{vmatrix} \frac{1}{{}^{\kappa}\mathbf{N}_{\Delta}} = \frac{{}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} \cdot {}^{\kappa}\mathbf{P}_{\backslash} - {}^{\kappa}\mathbf{N}_{\bullet\backslash} \cdot {}^{\kappa}\mathbf{P}_{\bullet\backslash}}{{}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} \cdot {}^{\kappa}\mathbf{N}_{\backslash} - ({}^{\kappa}\mathbf{N}_{\bullet\backslash})^2}$$

which may degenerate when ${}^{\kappa}\mathbf{N}_{\Delta} = 0$ to time-on-time

$$\begin{bmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{\kappa}\mathbf{k} \\ {}^{\kappa}\mathbf{h} \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ 0 \end{bmatrix} \qquad {}^{\kappa}\mathbf{k} = \frac{{}^{\kappa}\mathbf{P}_{\bullet\backslash}}{{}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}} \quad \text{and} \quad {}^{\kappa}\mathbf{h} = 0$$

Degeneracy occurs when the class is question has only appeared in a single race or in multiple races where $\mathbf{q}_r$ are the same or numerically indistinguishable. Because the $\mathbf{q}_r$ comes from the current state of the solver, degeneracy may be triggered for any class on any iteration; although, it becomes less likely with greater participation and effectively impossible if using the seeded data set.

This yields the vectors $\mathbf{k}$ and $\mathbf{h}$ which minimizes the performance index $\underline{Q}$ at the current value of $\mathbf{q}$ which remains fixed in this step of the algorithm. At this value of $\mathbf{q}$ there is no more optimization which can be done so we must improve the individual $\mathbf{q}_r$ to continue.

Pong discard the previous $\mathbf{q}$

For each r calculate a replacement value using the constant $\mathbf{N}$, $\overline{\mathbf{p}}$ and variable $\mathbf{k}$, $\mathbf{h}$

$$\mathbf{q}_r = \frac{{}^{\backslash\bullet}\mathbf{P}_r - {}^{\backslash\bullet\circ}\mathbf{N}_r}{{}^{\backslash\bullet\bullet}\mathbf{N}_r} = \frac{\sum_{\kappa} {}^{\kappa}\mathbf{k}\left[{}^{\kappa}\mathbf{P}_r - {}^{\kappa}\mathbf{h} \cdot {}^{\kappa}\mathbf{N}_r\right]}{{}^{\backslash\bullet\bullet}\mathbf{N}_r}$$

This yields the vector $\mathbf{q}$ which minimizes the performance index $\underline{Q}$ at the current value of $\mathbf{k}$ and $\mathbf{h}$ which remain fixed in this step of the algorithm. At this value of $\mathbf{k}$ and $\mathbf{h}$ there is no more optimization which can be done so we must improve the individual ${}^{\kappa}\mathbf{k}$ and ${}^{\kappa}\mathbf{h}$ to continue.

Continue while cost $\underline{Q}$ converges to a minimum.

18.3.2 The Algorithm for a Explicitly and Late Seeded Data Set

Staring with an initial guess for $\mathbf{q}_r$, $q_{\circledcirc}$, ${}^{\kappa}\mathbf{k}$ and ${}^{\kappa}\mathbf{h}$ each iteration of the ping-pong algorithm will come up with a lower value for the seeded performance index $\underline{Q}_{\circledcirc}$ which will converge on the final $\underline{\mathrm{SS}}_{\mathrm{E}}$

$$\underline{Q}_{\circledcirc} = \sum_r \sum_{\kappa} {}^{\kappa}\mathbf{N}_r \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k} \cdot \mathbf{q}_r - {}^{\kappa}\overline{\mathbf{p}}_r\right]^2 + \sum_{\kappa} \left[{}^{\kappa}\mathbf{h} + {}^{\kappa}\mathbf{k} \cdot \mathbf{q}_{\circledcirc}\right]^2$$

The performance index $\underline{Q}_{\circledcirc}$ is for the explicitly seeded data set where the ${}^{\kappa}\mathbf{N}_r$ and the ${}^{\kappa}\overline{\mathbf{p}}_r$ have been defined with respect to unseeded data. Note this is simply a different exposition for the same underlying algorithm as before.

With an initial value $\mathbf{q} = \overline{\overline{\mathbf{p}}}$ and $q_{\circledcirc} = 0$

Repeat

Ping discard the previous $\mathbf{k}$ and $\mathbf{h}$

For each κ calculate replacement values using the constant $\mathbf{N}$, $\overline{\mathbf{p}}$ and variable $\mathbf{q}$, $q_{\circledcirc}$

$$\begin{bmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}{+}q_{\circledcirc}^2 & {}^{\kappa}\mathbf{N}_{\bullet\backslash}{+}q_{\circledcirc} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash}\ {+}q_{\circledcirc} & {}^{\kappa}\mathbf{N}_{\backslash}\ {+}1 \end{bmatrix} \begin{bmatrix} {}^{\kappa}\mathbf{k} \\ {}^{\kappa}\mathbf{h} \end{bmatrix} = \begin{bmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{P}_{\backslash} \end{bmatrix} \quad \text{asserting} \quad {}^{\kappa}\mathbf{N}_{\Delta}^{\circledcirc} = \begin{vmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}{+}q_{\circledcirc}^2 & {}^{\kappa}\mathbf{N}_{\bullet\backslash}{+}q_{\circledcirc} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash}\ {+}q_{\circledcirc} & {}^{\kappa}\mathbf{N}_{\backslash}\ {+}1 \end{vmatrix} > 0$$

for solutions by Cramer's rule

$$
{}^{\kappa}\mathbf{k} = \begin{vmatrix} {}^{\kappa}\mathbf{P}_{\bullet\backslash} & {}^{\kappa}\mathbf{N}_{\bullet\backslash}+q_{\circledcirc} \\ {}^{\kappa}\mathbf{P}_{\backslash} & {}^{\kappa}\mathbf{N}_{\backslash}\;+1 \end{vmatrix} \frac{1}{{}^{\kappa}\mathbf{N}^{\circledcirc}_{\Delta}} = \frac{{}^{\kappa}\mathbf{P}_{\bullet\backslash}\,({}^{\kappa}\mathbf{N}_{\backslash}+1) - {}^{\kappa}\mathbf{P}_{\backslash}\,({}^{\kappa}\mathbf{N}_{\bullet\backslash}+q_{\circledcirc})}{({}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}+q^2_{\circledcirc})({}^{\kappa}\mathbf{N}_{\backslash}+1) - ({}^{\kappa}\mathbf{N}_{\bullet\backslash}+q_{\circledcirc})^2}
$$

$$
{}^{\kappa}\mathbf{h} = \begin{vmatrix} {}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}+q^2_{\circledcirc} & {}^{\kappa}\mathbf{P}_{\bullet\backslash} \\ {}^{\kappa}\mathbf{N}_{\bullet\backslash}\;+q_{\circledcirc} & {}^{\kappa}\mathbf{P}_{\backslash} \end{vmatrix} \frac{1}{{}^{\kappa}\mathbf{N}^{\circledcirc}_{\Delta}} = \frac{({}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}+q^2_{\circledcirc})\;{}^{\kappa}\mathbf{P}_{\backslash} - ({}^{\kappa}\mathbf{N}_{\bullet\backslash}+q_{\circledcirc})\;{}^{\kappa}\mathbf{P}_{\bullet\backslash}}{({}^{\kappa}\mathbf{N}_{\bullet\bullet\backslash}+q^2_{\circledcirc})({}^{\kappa}\mathbf{N}_{\backslash}+1) - ({}^{\kappa}\mathbf{N}_{\bullet\backslash}+q_{\circledcirc})^2}
$$

This yields the vectors $\mathbf{k}$ and $\mathbf{h}$ which minimizes the performance index $\underline{Q}_{\circledcirc}$ at the current value of $\mathbf{q}$ and $q_{\circledcirc}$ which remain fixed in this step of the algorithm. At this value of $\mathbf{q}$ and $q_{\circledcirc}$ there is no more optimization which can be done so we must improve the individual $\mathbf{q}_r$ and $q_{\circledcirc}$ to continue.

Pong discard the previous $\mathbf{q}$ and $q_{\circledcirc}$

For each r calculate a replacement value using the constant $\mathbf{N}$, $\overline{\mathbf{p}}$ and variable $\mathbf{k}$, $\mathbf{h}$

$$
q_r = \frac{{}^{\backslash\bullet}\mathbf{P}_r - {}^{\backslash\bullet\circ}\mathbf{N}_r}{{}^{\backslash\bullet\bullet}\mathbf{N}_r} = \frac{\sum_{\kappa} {}^{\kappa}\mathbf{k}\big[{}^{\kappa}\mathbf{P}_r - {}^{\kappa}\mathbf{h}\cdot{}^{\kappa}\mathbf{N}_r\big]}{{}^{\backslash\bullet\bullet}\mathbf{N}_r}
$$

For $\circledcirc$ calculate a replacement value using the variable $\mathbf{k}$ and $\mathbf{h}$

$$
q_{\circledcirc} = \frac{{}^{\backslash\bullet}\mathbf{P}_{\circledcirc} - {}^{\backslash\bullet\circ}\mathbf{N}_{\circledcirc}}{{}^{\backslash\bullet\bullet}\mathbf{N}_{\circledcirc}} = -\frac{\sum_{\kappa} {}^{\kappa}\mathbf{k}\cdot{}^{\kappa}\mathbf{h}}{\sum_{\kappa}({}^{\kappa}\mathbf{k})^2}
$$

This yields the vector $\mathbf{q}$ and the scalar $q_{\circledcirc}$ which minimizes the performance index $\underline{Q}_{\circledcirc}$ at the current value of $\mathbf{k}$ and $\mathbf{h}$ which remain fixed in this step of the algorithm. At this value of $\mathbf{k}$ and $\mathbf{h}$ there is no more optimization which can be done so we must improve the individual ${}^{\kappa}\mathbf{k}$ and ${}^{\kappa}\mathbf{h}$ to continue.

Continue while cost $\underline{Q}_{\circledcirc}$ converges to a minimum.

18.3.3 On the Assertion the Second-Factor Discriminant is Nonzero

On no realistic data set could the assertion ${}^{\kappa}\mathbf{N}^{\circledcirc}_{\Delta} > 0$ ever fail. The regressed $q_{\circledcirc}$ will always be outside the range of credible standard paces. It could be contrived with an artificial data set where half the fleet would necessarily get slower as the wind increases leading to a ridiculously large $q_{\circledcirc}$ which could then trigger degeneracy for a class that only appeared in a races with that same standard pace. It would never occur by accident.

Also note that we are not checking that the ${}^{\kappa}\mathbf{k}$ are all positive. This isn't necessary in any reasonably complete data set. Seeding the data set ensures this is the case for those classes that have only participated a few times.

18.3.4 On Stability of the Solutions

The algorithm throws away old values for the $\mathbf{q}$, $q_{\circledcirc}$, $\mathbf{k}$ and $\mathbf{h}$ as it goes along and will trigger large jumps for degenerate cases where ${}^{\kappa}\mathbf{N}_{\Delta}$ is zero and the ${}^{\kappa}\mathbf{k}$ and ${}^{\kappa}\mathbf{h}$ are underdetermined. Even solutions close to degeneracy will lead to numerical instability in the resulting ${}^{\kappa}\mathbf{k}$ and ${}^{\kappa}\mathbf{h}$. But this in no way effects the stable convergence of the $\underline{Q}$ so it is unclear whether this needs to be dealt with as the algorithm proceeds.

For the seeded case ${}^{\kappa}\mathbf{N}_{\Delta}$ (in its ${}^{\kappa}\mathbf{N}^{\circledcirc}_{\Delta}$ guise or otherwise) can be assured to be large enough to counter numerical degeneracy. In this case the ping-pong largely preserves the relative gauge; however a drifting gauge may cause the $\mathbf{q}_r$ to become small in magnitude which is bad for stability of the controls. For good numerical behaviour it may be reasonable to bump the gauge as the algorithm proceeds.

18.4 Second Order Terms in the Least Squares Solution

Part V

Running Statistics for Performance Handicaps

A running statistic is calculated by accumulating state in a single pass through the data.

Chapter 19

Baysian Statistics through Monte Carlo Simulations

Chapter 20

Recursive Least Squares

Chapter 21

Kalman Filters