Frank Systems of Coherent Electrical Units

In any rationalized system with a single unit of charge for both magnetic and electric phenomena

f	Lorentz Force Density
ρ	Charge Density
$\jmath$	Current Density

$$f = \rho E + \jmath \times B \qquad \nabla \times E + \frac{\partial}{\partial t} B = 0 \qquad \nabla \times H - \frac{\partial}{\partial t} D = \jmath$$

$$\nabla \cdot B = 0 \qquad \nabla \cdot D = \rho$$

With unitless relative permeability μ_r and unitless dielectric constant ε_r and with the constants of nature μ_0 and ε_0 (where permeability $\mu = \mu_r \mu_0$ and permittivity $\varepsilon = \varepsilon_r \varepsilon_0$) the Ampère and Coulomb Laws or (equivalently) the relation between the electromagnetic fields may be written

$$\frac{F_{\text{Ampère}}}{l} = \frac{\mu_r \mu_0 \imath I}{2\pi r} \quad \text{and} \quad F_{\text{Coulomb}} = \frac{qQ}{\varepsilon_r \varepsilon_0 4\pi r^2} \qquad \text{or} \qquad B = \mu_r \mu_0 H \quad \text{and} \quad E = \frac{1}{\varepsilon_r \varepsilon_0} D$$

Rewriting μ_0 and ε_0 in terms of the speed of light c and the vacuum impedance Z_0

$$\left. \begin{aligned} c &= \sqrt{\tfrac{1}{\mu_0 \varepsilon_0}} \\ Z_0 &= \sqrt{\tfrac{\mu_0}{\varepsilon_0}} \end{aligned} \right\} \iff \left\{ \begin{aligned} \mu_0 &= \tfrac{Z_0}{c} \\ \varepsilon_0 &= \tfrac{1}{c Z_0} \end{aligned} \right.$$

so that μ_0 and ε_0 may be eliminated from the laws

$$\frac{F_{\text{Ampère}}}{l} = \frac{\mu_r}{c} Z_0 \frac{\imath I}{2\pi r} \quad \text{and} \quad F_{\text{Coulomb}} = \frac{c}{\varepsilon_r} Z_0 \frac{qQ}{4\pi r^2} \qquad \text{or} \qquad \begin{aligned} B &= \tfrac{\mu_r}{c} Z_0 H \\ E &= \tfrac{c}{\varepsilon_r} Z_0 D \end{aligned}$$

In any coherent system TLMQ of fundamental time T, length L, mass M and charge Q

$$1\,\text{Derived Unit of Resistance} \triangleq \frac{1\,\text{Derived Unit of Action}}{1\,\text{Fundamental Unit of Charge}^2} = 1\,\text{Q}^{-2}\text{ML}^2\text{T}^{-1}$$

When the unit of charge is chosen so that $Z_0 = 1\,\text{Derived Unit of Resistance} = 1\,\text{Q}^{-2}\text{ML}^2\text{T}^{-1}$ then Q is effectively a derived unit while maintaining the form of a fundamental unit — this leads to straightforward conversions to and from other systems of electromagnetic units. Formally setting $Z_0 = 1$ then leads to an equivalent but incommensurable TLM($Z_0 \triangleq 1$) system with unitless resistance, a derived unit of charge $\text{M}^{1/2}\text{LT}^{-1/2} = \sqrt{1\,\text{Derived Unit of Action}}$, and having a streamlined presentation of the Ampère and Coulomb Laws

$$\frac{F_{\text{Ampère}}}{l} = \frac{\mu_r}{c} \frac{\imath I}{2\pi r} \quad \text{and} \quad F_{\text{Coulomb}} = \frac{c}{\varepsilon_r} \frac{qQ}{4\pi r^2} \qquad \text{or} \qquad \begin{aligned} B &= \tfrac{\mu_r}{c} H \\ E &= \tfrac{c}{\varepsilon_r} D \end{aligned}$$

Given a precursor system of units TLM*Q where the vacuum impedance Z_0 does not evaluate to $1\,{}^\star\text{Q}^{-2}\text{ML}^2\text{T}^{-1}$ there is a commensurable unit of charge Q that can be computed directly so that

$$1\,\text{Q} \triangleq \sqrt{\frac{1\,\text{Derived Unit of Action}}{Z_0}} \equiv \sqrt{\frac{1\,\text{ML}^2\text{T}^{-1}}{Z_0}}$$

That is to say, letting ζ_0 be the numerical value of the constant Z_0 in the T, L, M, *Q units

$$\text{(algebraically)} \qquad Z_0 = \zeta_0\,{}^\star\text{Q}^{-2}\text{ML}^2\text{T}^{-1} \iff \zeta_0 \triangleq \frac{Z_0}{1\,{}^\star\text{Q}^{-2}\text{ML}^2\text{T}^{-1}}$$

gives the conversion

$$\frac{1\,\text{Q}}{1\,{}^\star\text{Q}} = \frac{1}{\sqrt{\zeta_0}}$$

and a fundamental unit of charge Q corresponding to the derived unit of charge $\text{M}^{1/2}\text{LT}^{-1/2}$.

SI (prior to the 2019 rebasing) had exact constants

$$\mu_0/4\pi = 10^{-7}\,\mathrm{H/m} \qquad c = 299\,792\,458\,\mathrm{m/s} \qquad Z_0/4\pi = 29.979\,245\,8\,\Omega$$

or approximately

$$c \approx 300\,\mathrm{m/\mu s} \quad Z_0 \approx 377\,\Omega \quad c^{-1} \approx 3.33\,\mathrm{ns/m}$$

The *frankcoulomb* FC — what the SI unit of charge *should have been* — is the unit of charge (via an exact conversion prior to the quantization implicit in the 2019 rebasing)

$$1\,\mathrm{FC} = \sqrt{\frac{1\,\mathrm{J{\cdot}s}}{Z_0}} = \frac{1}{\sqrt{4\pi\times 29.979\,245\,8}}\,\mathrm{C}$$

for which $Z_0 = 1\,\mathrm{J{\cdot}s/FC^2}$ and

$$\sqrt{1\,\mathrm{J{\cdot}s}} = 1\,\mathrm{kg^{1/2}{\cdot}m{\cdot}s^{-1/2}}$$

is the equivalent derived unit. Other coherent electrical units are derived with respect to this unit via proportions

$$\begin{aligned}
\bullet^{-2} &= 4\pi\times 29.979\,245\,8 \approx 377\\
\bullet^{-1} &= \sqrt{4\pi\times 29.979\,245\,8} \approx 19.4\\
\bullet &= \frac{1}{\sqrt{4\pi\times 29.979\,245\,8}} \approx \frac{1}{19.4} \approx 51.5\times 10^{-3}\\
\bullet^{2} &= \frac{1}{4\pi\times 29.979\,245\,8} \approx \frac{1}{377} \approx 2.65\times 10^{-3}
\end{aligned}$$

which convert SI to the SMKF — *second-metre-kilogramme-frankcoulomb* — system or the equivalent but incommensurable SMK($Z_0 \triangleq 1$) system according to the power of Q in the TLMQ — dimensionality of time, length, mass and charge respectively — column in the table below. Having done so, the corresponding unit in any other TLM($Z_0 \triangleq 1$) system can be determined using the final column of the table using only the conversions for time, length and mass from SMK.

measurement		frank unit	conversion		∝	TLMQ	TLM($Z_0 \triangleq 1$)
electric charge	Q	frankcoulomb	FC	51.5 mC	$\bullet$	Q	$M^{1/2}LT^{-1/2}$
electric current	I	frankamp	FA	51.5 mA	$\bullet$	QT^{-1}	$M^{1/2}LT^{-3/2}$
voltage	V	frankvolt	FV	19.4 V	$\bullet^{-1}$	$Q^{-1}ML^2T^{-2}$	$M^{1/2}LT^{-3/2}$
capacitance	C	frankfarad	FF	2.65 mF	$\bullet^{2}$	$Q^2M^{-1}L^{-2}T^2$	T
conductance	G	franksiemens	FS	2.65 mS	$\bullet^{2}$	$Q^2M^{-1}L^{-2}T$	1
conductivity	σ		FS/m	2.65 mS/m	$\bullet^{2}$	$Q^2M^{-1}L^{-3}T$	L^{-1}
resistance	R	frankohm	FΩ	377 Ω	$\bullet^{-2}$	$Q^{-2}ML^2T^{-1}$	1
inductance	L	frankhenry	FH	377 H	$\bullet^{-2}$	$Q^{-2}ML^2$	T
displacement field	D		FC/m²	51.5 mC/m²	$\bullet$	QL^{-2}	$M^{1/2}L^{-1}T^{-1/2}$
magnetizing field	H		FA/m	51.5 mA/m	$\bullet$	$QL^{-1}T^{-1}$	$M^{1/2}T^{-3/2}$
electric field	E		FV/m	19.4 V/m	$\bullet^{-1}$	$Q^{-1}MLT^{-2}$	$M^{1/2}L^{-1}T^{-1/2}$
magnetic field	B	franktesla	FT	19.4 T	$\bullet^{-1}$	$Q^{-1}MT^{-1}$	$M^{1/2}L^{-1}T^{-1/2}$
electric flux	Φ_E		FV·m	19.4 V·m	$\bullet^{-1}$	$Q^{-1}ML^3T^{-2}$	$M^{1/2}L^2T^{-3/2}$
magnetic flux	Φ_B	frankweber	FWb	19.4 Wb	$\bullet^{-1}$	$Q^{-1}ML^2T^{-1}$	$M^{1/2}LT^{-1/2}$
permittivity	ε		FF/m	2.65 mF/m	$\bullet^{2}$	$Q^2M^{-1}L^{-3}T^2$	$L^{-1}T$
permeability	μ		FH/m	377 H/m	$\bullet^{-2}$	$Q^{-2}ML$	$L^{-1}T$

In frank (and derived) units

$$c = 299\,792\,458\,\mathrm{m/s} \qquad \mu_0 = 1/299\,792\,458\,\mathrm{FH/m}\ (\mathrm{s/m})$$
$$Z_0 = 1\,\mathrm{F\Omega}\ (\text{unitless}) \qquad \varepsilon_0 = 1/299\,792\,458\,\mathrm{FF/m}\ (\mathrm{s/m})$$

Note that the franksiemens and frankohm, being directly derived from the impedance of free space, are invariant across all systems of coherent TLM units. And the frankhenry and frankfarad are common to all coherent systems of units based on the second.