Comparison of Derived Units of Charge

Ampère's and Coulomb's Laws in a vacuum

$$\frac{F_{\text{Ampère}}}{l} = \frac{\mu_0 \cdot \imath I}{2\pi r} \equiv Z_0 \frac{\imath I}{c \cdot 2\pi r} \qquad F_{\text{Coulomb}} = \frac{\kappa_0 \cdot qQ}{4\pi r^2} \equiv Z_0 \frac{c \cdot qQ}{4\pi r^2} \equiv \frac{qQ}{\varepsilon_0 \cdot 4\pi r^2}$$

The vacuum permeability μ_0, the impedance of free space Z_0 and the Coulomb constant κ_0 (reciprocal to the vacuum permittivity ε_0) form a geometric sequence with a factor of c, the speed of light, between terms

$$\begin{bmatrix}\mu_0 & Z_0 & \kappa_0\end{bmatrix} = \mu_0 \begin{bmatrix}1 & c & c^2\end{bmatrix} = \begin{bmatrix}\mu_0 & \mu_0 c & \mu_0 c^2\end{bmatrix}$$

Any two of c, μ_0, Z_0 and κ_0 are sufficient to determine the others. These *rationalized* constants are scaled from obsolescent variants by a factor of 4π. We distinguish these obsolescent forms with a $\star$ superscript: $4\pi\mu_0^\star = \mu_0$, $4\pi Z_0^\star = Z_0$ and $4\pi\kappa_0^\star = \kappa_0$ (or reciprocally, $\varepsilon_0^\star = 4\pi\varepsilon_0$). Ampere's law in this obsolescent mode has a peculiar factor of two where the 4π scaling interacts with the $2\pi r$ circumferential term.

$$\frac{F_{\text{Ampère}}}{l} = \frac{2\mu_0^\star \cdot \imath I}{r} \equiv 2Z_0^\star \frac{\imath I}{c \cdot r} \qquad F_{\text{Coulomb}} = \frac{\kappa_0^\star \cdot qQ}{r^2} \equiv Z_0^\star \frac{c \cdot qQ}{r^2} \equiv \frac{qQ}{\varepsilon_0^\star \cdot r^2}$$

In any coherent TLM system of units we have a $\sqrt{\frac{1\,\mathrm{LT^{-1}}}{c}}$ geometric sequence of charge units

$$\begin{bmatrix}1\,\text{ab.u} & 1\,\text{frank.u} & 1\,\text{stat.u}\end{bmatrix} = \begin{bmatrix}\sqrt{\frac{1\,\mathrm{ML}}{\mu_0}} & \sqrt{\frac{1\,\mathrm{ML^2T^{-1}}}{Z_0}} & \sqrt{\frac{1\,\mathrm{ML^3T^{-2}}}{\kappa_0}}\end{bmatrix}$$

$$= \tfrac{1}{\sqrt{4\pi}} \begin{bmatrix}1\,\text{ab.u}_\star & 1\,\text{frank.u}_\star & 1\,\text{stat.u}_\star\end{bmatrix} = \tfrac{1}{\sqrt{4\pi}} \begin{bmatrix}\sqrt{\frac{1\,\mathrm{ML}}{\mu_0^\star}} & \sqrt{\frac{1\,\mathrm{ML^2T^{-1}}}{Z_0^\star}} & \sqrt{\frac{1\,\mathrm{ML^3T^{-2}}}{\kappa_0^\star}}\end{bmatrix}$$

In terms of a traditional coulomb (exactly prior to the SI rebasing of 2019)

$$\mu_0^\star = 10^{-2}\,\mathrm{dyn{\cdot}s^2/C^2} \equiv 10^{-7}\,\mathrm{N{\cdot}s^2/C^2} \qquad c = 29\,979\,245\,800\,\mathrm{cm/s} \equiv 299\,792\,458\,\mathrm{m/s}$$

For the oldest second-centimetre-gramme units with a factor of $1/\sqrt{29\,979\,245\,800}$ between terms

$$1\,\text{ab.coul}_\star = \sqrt{\frac{1\,\mathrm{dyn{\cdot}s^2}}{\mu_0^\star}} = \sqrt{10^2}\,\mathrm{C} = 10\,\mathrm{C} \quad \text{resulting in} \quad \mu_0^\star = 1\,\text{ab.coul}_\star^{-2}{\cdot}\mathrm{g{\cdot}cm}$$

$$1\,\text{frank.coul}_\star = \sqrt{\frac{1\,\mathrm{dyn{\cdot}cm{\cdot}s}}{Z_0^\star}} = \frac{10}{\sqrt{29\,979\,245\,800}}\,\mathrm{C} \approx 57.8\,\mu\mathrm{C} \quad \text{resulting in} \quad Z_0^\star = 1\,\text{frank.coul}_\star^{-2}{\cdot}\mathrm{g{\cdot}cm^2{\cdot}s^{-1}}$$

$$1\,\text{stat.coul}_\star = \sqrt{\frac{1\,\mathrm{dyn{\cdot}cm^2}}{\kappa_0^\star}} = \frac{10}{29\,979\,245\,800}\,\mathrm{C} \approx 334\,\mathrm{pC} \quad \text{resulting in} \quad \kappa_0^\star = 1\,\text{stat.coul}_\star^{-2}{\cdot}\mathrm{g{\cdot}cm^3{\cdot}s^{-2}}$$

The rationalized units are reduced by a factor of $\sqrt{4\pi} \approx 3\frac{6}{11}$ from their obsolescent counterparts

$$1\,\text{ab.coul} = \sqrt{\frac{1\,\mathrm{dyn{\cdot}s^2}}{\mu_0}} = \frac{10}{\sqrt{4\pi}}\,\mathrm{C} \approx 2.82\,\mathrm{C} \quad \text{resulting in} \quad \mu_0 = 1\,\text{ab.coul}^{-2}{\cdot}\mathrm{g{\cdot}cm}$$

$$1\,\text{frank.coul} = \sqrt{\frac{1\,\mathrm{dyn{\cdot}cm{\cdot}s}}{Z_0}} = \frac{10}{\sqrt{29\,979\,245\,800 \times 4\pi}}\,\mathrm{C} \approx 16.3\,\mu\mathrm{C} \quad \text{resulting in} \quad Z_0 = 1\,\text{frank.coul}^{-2}{\cdot}\mathrm{g{\cdot}cm^2{\cdot}s^{-1}}$$

$$1\,\text{stat.coul} = \sqrt{\frac{1\,\mathrm{dyn{\cdot}cm^2}}{\kappa_0}} = \frac{10}{29\,979\,245\,800 \times \sqrt{4\pi}}\,\mathrm{C} \approx 94.1\,\mathrm{pC} \quad \text{resulting in} \quad \kappa_0 = 1\,\text{stat.coul}^{-2}{\cdot}\mathrm{g{\cdot}cm^3{\cdot}s^{-2}}$$

For unhistorical second-metre-kilogramme units with a factor of $1/\sqrt{299\,792\,458}$ between terms

$$1\,\text{Ab.Coul}_\star = \sqrt{\frac{1\,\mathrm{N{\cdot}s^2}}{\mu_0^\star}} = \sqrt{10^7}\,\mathrm{C} \approx 3.16\,\mathrm{kC} \quad \text{resulting in} \quad \mu_0^\star = 1\,\text{Ab.Coul}_\star^{-2}{\cdot}\mathrm{kg{\cdot}m}$$

$$1\,\text{Frank.Coul}_\star = \sqrt{\frac{1\,\mathrm{N{\cdot}m{\cdot}s}}{Z_0^\star}} = \frac{\sqrt{10^7}}{\sqrt{299\,792\,458}}\,\mathrm{C} \approx 193\,\mathrm{mC} \quad \text{resulting in} \quad Z_0^\star = 1\,\text{Frank.Coul}_\star^{-2}{\cdot}\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}$$

$$1\,\text{Stat.Coul}_\star = \sqrt{\frac{1\,\mathrm{N{\cdot}m^2}}{\kappa_0^\star}} = \frac{\sqrt{10^7}}{299\,792\,458}\,\mathrm{C} \approx 10.5\,\mu\mathrm{C} \quad \text{resulting in} \quad \kappa_0^\star = 1\,\text{Stat.Coul}_\star^{-2}{\cdot}\mathrm{kg{\cdot}m^3{\cdot}s^{-2}}$$

And with the most modern rationalized second-metre-kilogramme base

$$1\,\text{Ab.Coul} = \sqrt{\frac{1\,\mathrm{N{\cdot}s^2}}{\mu_0}} = \frac{\sqrt{10^7}}{\sqrt{4\pi}}\,\mathrm{C} \approx 892\,\mathrm{C} \quad \text{resulting in} \quad \mu_0 = 1\,\text{Ab.Coul}^{-2}{\cdot}\mathrm{kg{\cdot}m}$$

$$1\,\text{Frank.Coul} = \sqrt{\frac{1\,\mathrm{N{\cdot}m{\cdot}s}}{Z_0}} = \frac{\sqrt{10^7}}{\sqrt{299\,792\,458 \times 4\pi}}\,\mathrm{C} \approx 51.5\,\mathrm{mC} \quad \text{resulting in} \quad Z_0 = 1\,\text{Frank.Coul}^{-2}{\cdot}\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}$$

$$1\,\text{Stat.Coul} = \sqrt{\frac{1\,\mathrm{N{\cdot}m^2}}{\kappa_0}} = \frac{\sqrt{10^7}}{299\,792\,458 \times \sqrt{4\pi}}\,\mathrm{C} \approx 2.98\,\mu\mathrm{C} \quad \text{resulting in} \quad \kappa_0 = 1\,\text{Stat.Coul}^{-2}{\cdot}\mathrm{kg{\cdot}m^3{\cdot}s^{-2}}$$

Dimensionality of Electromagnetic Units in Terms of Fundamental Units

The rationalized and obsolescent systems of units have identical dimensionality so the table below works for either form. TLMQI are the dimensionality of time, length, mass, charge and current respectively. In TLMQ the unit of current I is replaced with the derived unit $\mathrm{QT^{-1}}$. TLM[ab.u], TLM[frank.u] and TLM[stat.u] are instances of TLMQ which are equivalent to TLM($\mu_0 \triangleq 1$), TLM($Z_0 \triangleq 1$) and TLM($\varepsilon_0 \triangleq 1$) where the unit of charge is derived so that the corresponding constant is eliminated. Gaussian-style units substitute the TLM($\varepsilon_0 \triangleq 1$) unit of charge and the TLM($\mu_0 \triangleq 1$) unit of current into the TLMQI form.

<table>
<tr><th colspan="2">measurement</th><th>TLMQI</th><th>TLMQ</th><th>TLM($\mu_0 \triangleq 1$)</th><th>TLM($Z_0 \triangleq 1$)</th><th>TLM($\varepsilon_0 \triangleq 1$)</th><th>Gaussian</th></tr>
<tr><td>electric charge</td><td>Q</td><td colspan="2">Q</td><td>$\mathrm{M^{1/2}L^{1/2}}$</td><td>$\mathrm{M^{1/2}LT^{-1/2}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-1}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-1}}$</td></tr>
<tr><td>electric current</td><td>I</td><td>I</td><td>$\mathrm{QT^{-1}}$</td><td>$\mathrm{M^{1/2}L^{1/2}T^{-1}}$</td><td rowspan="2">$\mathrm{M^{1/2}LT^{-3/2}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-2}}$</td><td rowspan="2">$\mathrm{M^{1/2}L^{1/2}T^{-1}}$</td></tr>
<tr><td>voltage</td><td>V</td><td colspan="2">$\mathrm{Q^{-1}ML^2T^{-2}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-2}}$</td><td>$\mathrm{M^{1/2}L^{1/2}T^{-1}}$</td></tr>
<tr><td>capacitance</td><td>C</td><td colspan="2">$\mathrm{Q^2M^{-1}L^{-2}T^2}$</td><td>$\mathrm{L^{-1}T^2}$</td><td>T</td><td>L</td><td>L</td></tr>
<tr><td>conductance</td><td>G</td><td>$\mathrm{IQM^{-1}L^{-2}T^2}$</td><td>$\mathrm{Q^2M^{-1}L^{-2}T}$</td><td>$\mathrm{L^{-1}T}$</td><td>1</td><td>$\mathrm{LT^{-1}}$</td><td>1</td></tr>
<tr><td>conductivity</td><td>σ</td><td>$\mathrm{IQM^{-1}L^{-3}T^2}$</td><td>$\mathrm{Q^2M^{-1}L^{-3}T}$</td><td>$\mathrm{L^{-2}T}$</td><td>$\mathrm{L^{-1}}$</td><td>$\mathrm{T^{-1}}$</td><td>$\mathrm{L^{-1}}$</td></tr>
<tr><td>resistance</td><td>R</td><td>$\mathrm{I^{-1}Q^{-1}ML^2T^{-2}}$</td><td>$\mathrm{Q^{-2}ML^2T^{-1}}$</td><td>$\mathrm{LT^{-1}}$</td><td>1</td><td>$\mathrm{L^{-1}T}$</td><td>1</td></tr>
<tr><td>inductance</td><td>L</td><td>$\mathrm{I^{-1}Q^{-1}ML^2T^{-1}}$</td><td>$\mathrm{Q^{-2}ML^2}$</td><td>L</td><td>T</td><td>$\mathrm{L^{-1}T^2}$</td><td>T</td></tr>
<tr><td>displacement field</td><td>D</td><td colspan="2">$\mathrm{QL^{-2}}$</td><td>$\mathrm{M^{1/2}L^{-3/2}}$</td><td>$\mathrm{M^{1/2}L^{-1}T^{-1/2}}$</td><td rowspan="2">$\mathrm{M^{1/2}L^{-1/2}T^{-1}}$</td><td rowspan="4">$\mathrm{M^{1/2}L^{-1/2}T^{-1}}$</td></tr>
<tr><td>electric field</td><td>E</td><td colspan="2">$\mathrm{Q^{-1}MLT^{-2}}$</td><td>$\mathrm{M^{1/2}L^{1/2}T^{-2}}$</td><td rowspan="2">$\mathrm{M^{1/2}T^{-3/2}}$</td></tr>
<tr><td>magnetizing field</td><td>H</td><td>$\mathrm{IL^{-1}}$</td><td>$\mathrm{QL^{-1}T^{-1}}$</td><td rowspan="2">$\mathrm{M^{1/2}L^{-1/2}T^{-1}}$</td><td>$\mathrm{M^{1/2}L^{1/2}T^{-2}}$</td></tr>
<tr><td>magnetic field</td><td>B</td><td>$\mathrm{I^{-1}MT^{-2}}$</td><td>$\mathrm{Q^{-1}MT^{-1}}$</td><td>$\mathrm{M^{1/2}L^{-1}T^{-1/2}}$</td><td>$\mathrm{M^{1/2}L^{-3/2}}$</td></tr>
<tr><td>electric flux</td><td>Φ_E</td><td colspan="2">$\mathrm{Q^{-1}ML^3T^{-2}}$</td><td>$\mathrm{M^{1/2}L^{5/2}T^{-2}}$</td><td>$\mathrm{M^{1/2}L^2T^{-3/2}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-1}}$</td><td rowspan="2">$\mathrm{M^{1/2}L^{3/2}T^{-1}}$</td></tr>
<tr><td>magnetic flux</td><td>Φ_B</td><td>$\mathrm{I^{-1}ML^2T^{-2}}$</td><td>$\mathrm{Q^{-1}ML^2T^{-1}}$</td><td>$\mathrm{M^{1/2}L^{3/2}T^{-1}}$</td><td>$\mathrm{M^{1/2}LT^{-1/2}}$</td><td>$\mathrm{M^{1/2}L^{1/2}}$</td></tr>
<tr><td>permittivity</td><td>ε</td><td colspan="2">$\mathrm{Q^2M^{-1}L^{-3}T^2}$</td><td>$\mathrm{L^{-2}T^2}$</td><td rowspan="2">$\mathrm{L^{-1}T}$</td><td>1</td><td rowspan="2">1</td></tr>
<tr><td>permeability</td><td>μ</td><td>$\mathrm{I^{-2}MLT^2}$</td><td>$\mathrm{Q^{-2}ML}$</td><td>1</td><td>$\mathrm{L^{-2}T^2}$</td></tr>
</table>

In SI the rebasing of 2019 defines a coulomb exactly in terms of the charge of the electron, only approximates its previous value, and cannot be considered a derived unit at all. Even prior to the 2019 rebasing, SI had not formally used a derived unit of charge — that the SI coulomb had been defined to be exactly one-tenth of the Gaussian abcoulomb [ab.coul$_\star$] occurred almost in passing; the derivation had never been particularly useful and had been hidden in the numeric form of μ_0 as $4\pi \times 10^{-7}\ \mathrm{H/m}$.

Not in SI but the unit of charge you should have expected (prior to the rebasing of 2019) would have been the second-metre-kilogramme rationalized frankcoulomb (heretofore abbreviated [Frank.Coul] for comparison purposes but hereafter written [FC] for brevity)

$$1\,\mathrm{FC} = \sqrt{\frac{1\,\mathrm{N{\cdot}m{\cdot}s}}{Z_0}} \overset{\text{(exactly prior to 2019)}}{=} \frac{\sqrt{10^7}}{\sqrt{299\,792\,458 \times 4\pi}}\,\mathrm{C} \approx 51.5\,\mathrm{mC}$$

That is, you would have expected SI to have a suite of electromagnetic units where the frankhenry [FH], frankohm [FΩ] and frankfarad [FF] are defined such that

$$\mu_0 = \frac{1}{299\,792\,458}\,\mathrm{FH/m} = 1\,\mathrm{FH/lightsecond} \qquad Z_0 = 1\,\mathrm{F\Omega} \qquad \varepsilon_0 = \frac{1}{299\,792\,458}\,\mathrm{FF/m} = 1\,\mathrm{FF/lightsecond}$$

Arthur Kennelly championed such a unit of charge but it was never used practically; this is a pity as it leads to a very convenient and neat system of electrical and magnetic units. Here it is presented in the TLM[frank.u] form so that the unit of charge is nominally specified as a fundamental unit which just so happens to force extremely convenient numerical expressions of the fundamental constants.